Biomedical Ultrasonics, Biotherapy & Biopharmaceuticals Laboratory

Short Course on Nonlinear Acoustics
Part I

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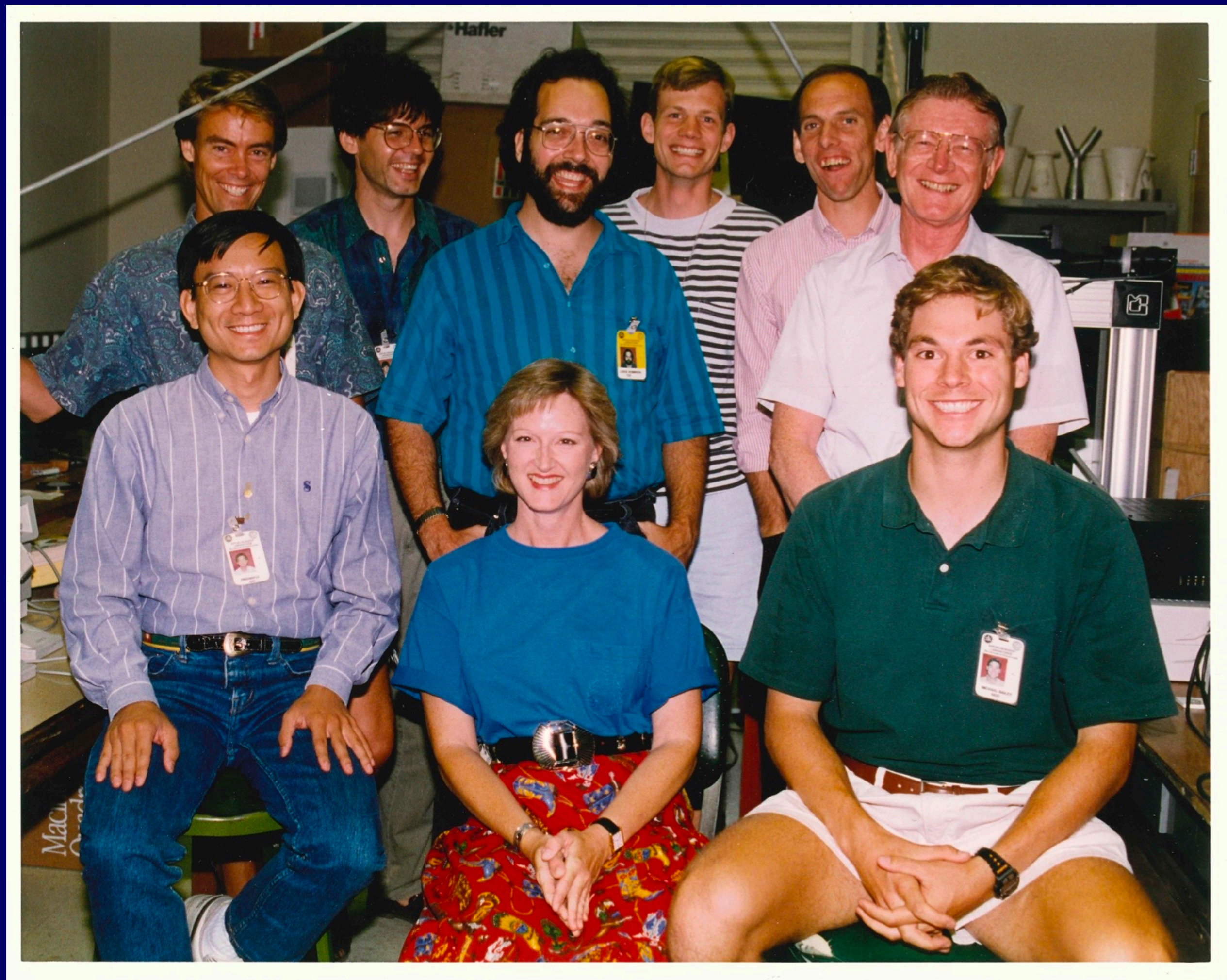
Outline

1. Nonlinearity
2. Distortion and Harmonic generation
3. Shock formation
4. Weak shocks
5. Burgers Equation
6. Taylor shock thickness
7. Diffraction effects: Westervelt and KZK equation

David T Blackstock (1930-2021)



Fluid Dynamics Equations

BUBBL

Conservation of Mass (continuity)

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = \mathbf{0}$$

Conservation of Momentum (Compressible Navier Stokes)

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla \mathbf{P} + \mu \nabla \cdot \nabla \mathbf{u} + \left(\mu_{\mathrm{B}} + \frac{\mu}{3}\right) \nabla (\nabla \cdot \mathbf{u})$$

Equation of State

$$P = P(\rho, s)$$

Thermodynamics

$$\rho \frac{\partial s}{\partial t} = \kappa \nabla^2 T + loss$$

P	Total pressure
ρ	Density
$\boldsymbol{u}$	Particle velocity
μ	Shear viscosity
μ_B	Bulk viscosity
s	Entropy
κ	Thermal conductivity
T	Temperature

Finite-Amplitude Acoustics

BUBBL

Sound wave is a perturbation on background properties

$$P = p_0 + p'$$
$$\rho = \rho_0 + \rho'$$
$$\mathbf{u} = 0 + \mathbf{u}$$

Conservation of Mass (continuity)

$$\cancel{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} + \rho' \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho' + \cancel{\mathbf{u} \cdot \nabla \rho_0} = 0$$

Conservation of Momentum (Compressible Navier Stokes)

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \rho' \frac{\partial \mathbf{u}}{\partial t} + \rho_0 \mathbf{u} \cdot \nabla \mathbf{u} + \rho' \mathbf{u} \cdot \nabla \mathbf{u} = \cancel{-\nabla p_0} - \nabla p' + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla (\nabla \cdot \mathbf{u})$$

Acoustic Mach Number

From linear acoustics

$$\frac{u}{c_0} = \frac{p'}{\rho_0 c_0^2} = \frac{\rho'}{\rho_0}$$

Plane wave   Always

Acoustic Mach number

$$\epsilon = \frac{u_0}{c_0}$$

Characteristic velocity

Small when $u_0 \ll c_0$

$$p_a \ll \rho_0 c_0^2$$

$$\cancel{p_a \ll p_0}$$

Ideal Gas:

$$p_a \ll \gamma p_0$$

$$\epsilon = 10^{-2}$$

Air: 1400 Pa (154 dB re 20 uPa)
Water: 22 MPa (264 dB re 1 uPa)

Second Order Wave Equation

Keeping up to second order terms

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2} - \frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial t^3} - \left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho_0 u^2}{2} - \frac{p^2}{2\rho_0 c_0^2}\right)$$

Naze Tjøtta and Tjøtta JASA 69:1644 (1981).

≈ 0 Lagrangian Density

Westervelt JASA (1963).

p acoustic pressure
$c0$ small signal sound speed
$\rho 0$ density
β coefficient of nonlinearity
δ diffusivity of sound; thermal conduction and viscosity
u particle velocity

Continuity

β = 1+ B/2A

State

1.2 air
3.5 water
5 tissue

Lossless Progressive Plane Waves

Progressive:

$$\frac{\partial p}{\partial x} - \frac{1}{c_0}\frac{\partial p}{\partial t} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial t} + \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial t^2}$$

Lossless:

$$\frac{\partial p}{\partial x} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau}$$

Retarded time

$$\tau = t - \frac{x}{c_0}$$

Nondimensionalise

$$P = p / p_0$$

$$\theta = \omega\tau$$

Characteristic pressure

Characteristic frequency

$$\frac{\partial P}{\partial x} = \frac{\beta\omega p_0}{\rho_0 c_0^3} P \frac{\partial P}{\partial \theta}$$

Poisson Solution

$$P(\theta) = f\left(\theta + \frac{\beta \omega p_0}{\rho_0 c_0^3} x f(\theta)\right)$$
$$= f\left(\theta + \sigma f(\theta)\right)$$
$$P\left(\theta' - \sigma f(\theta)\right) = f(\theta')$$

$$\sigma = \frac{\beta p_0 \omega x}{\rho_0 c_0^3} = \beta \varepsilon k x$$

Acoustic Mach Number

Matlab

```
%Poisson solution
theta=linspace(-pi,pi,30);
ps=sin(theta);

sigma=1;

tdistort=theta-sigma*ps;

plot(tdistort,ps,’o',theta,ps,’.');
```

Excel

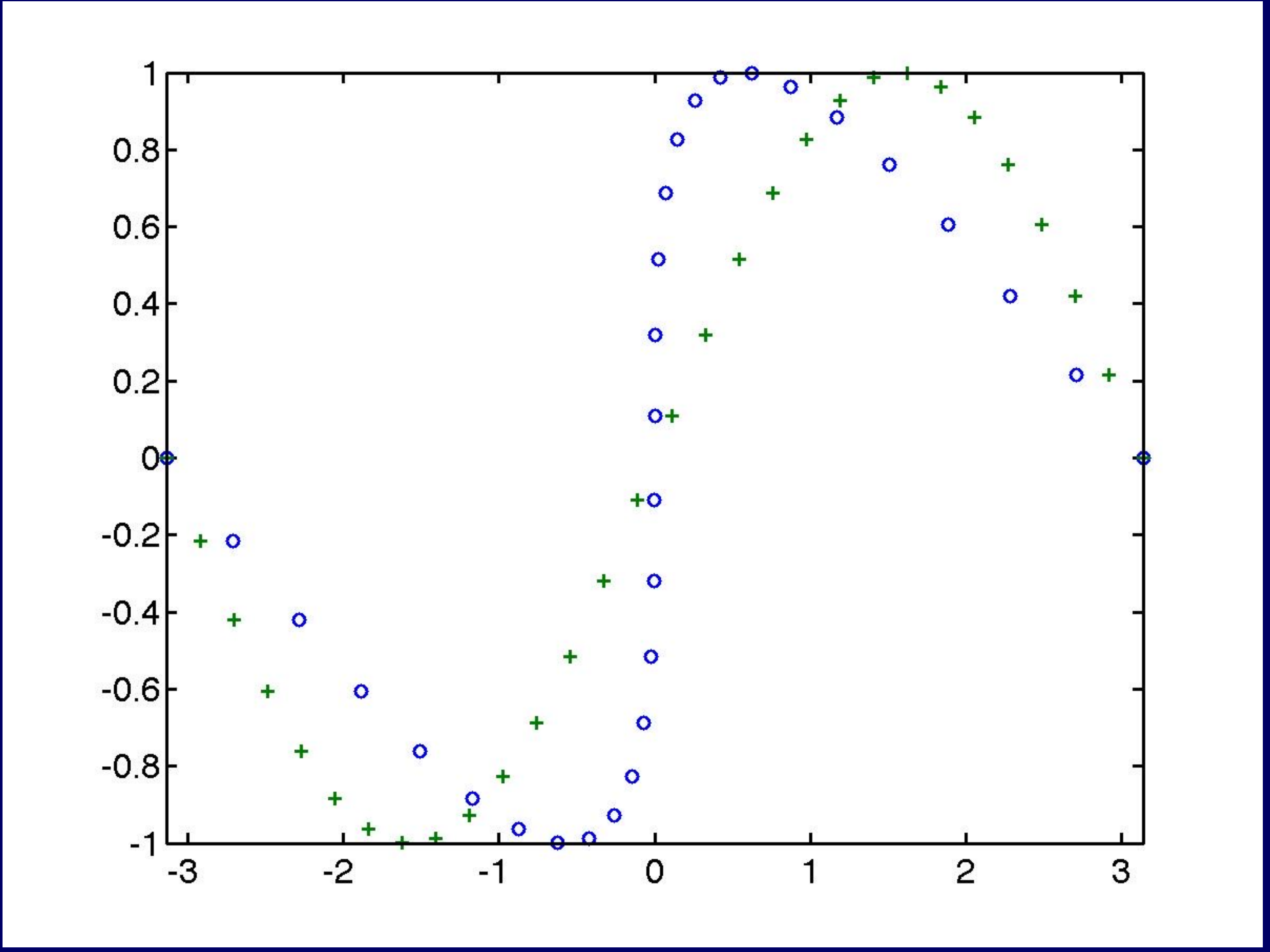
	A	B	C
1	Numpts		Sigma
2	30		1
3			
4	theta	ps	tdistort
5	-3.1415927	-1.225E-16	-3.1415927
6	-2.9321531	-0.2079117	-2.7242415
7	-2.7227136	-0.4067366	-2.315977
8	-2.5132741	-0.5877853	-1.9254889

=A5+4*ASIN(1)/A$2

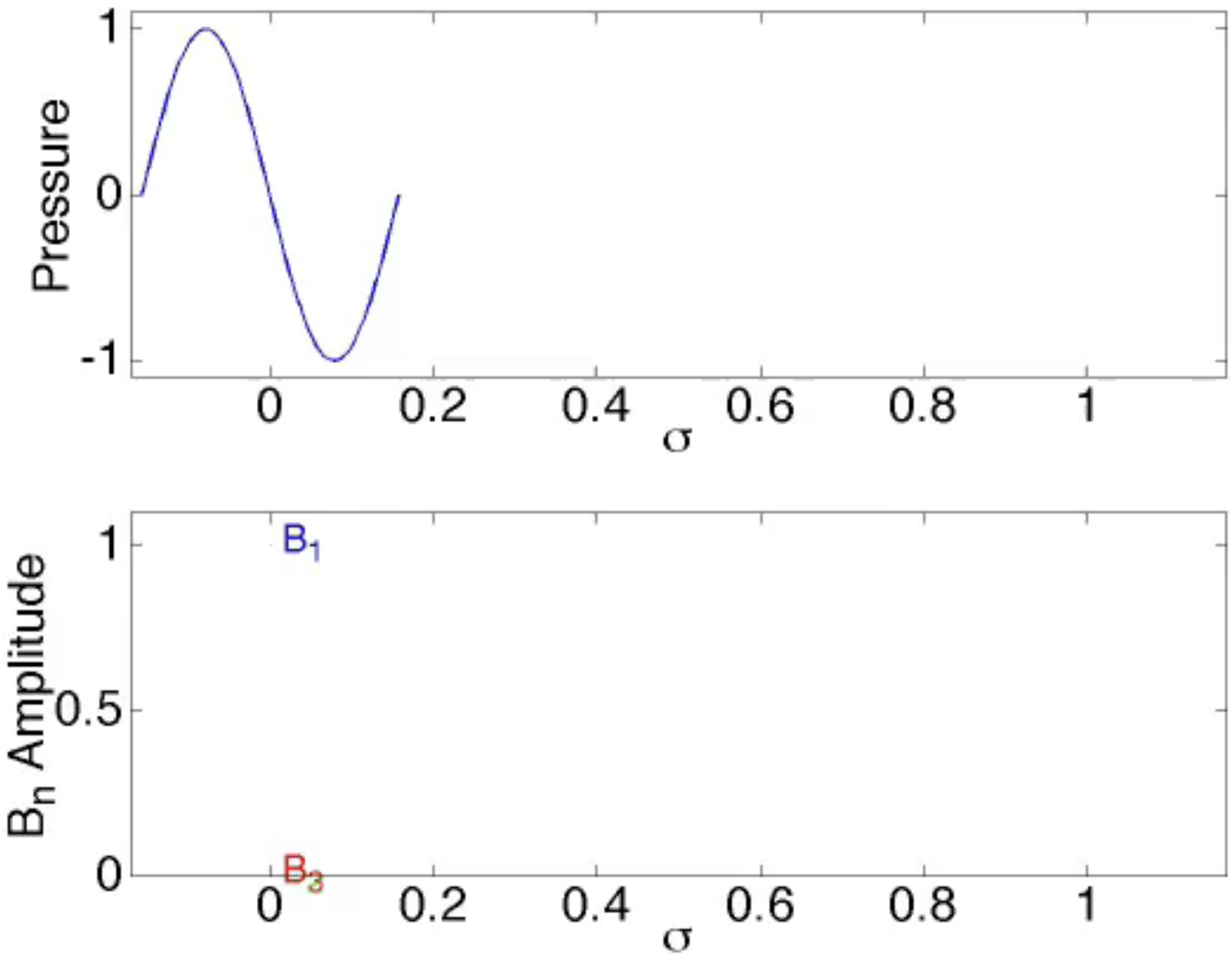
=SIN(A6)

=A6-C$2*B6

Poisson Solution



Harmonic Generation



Fubini Solution

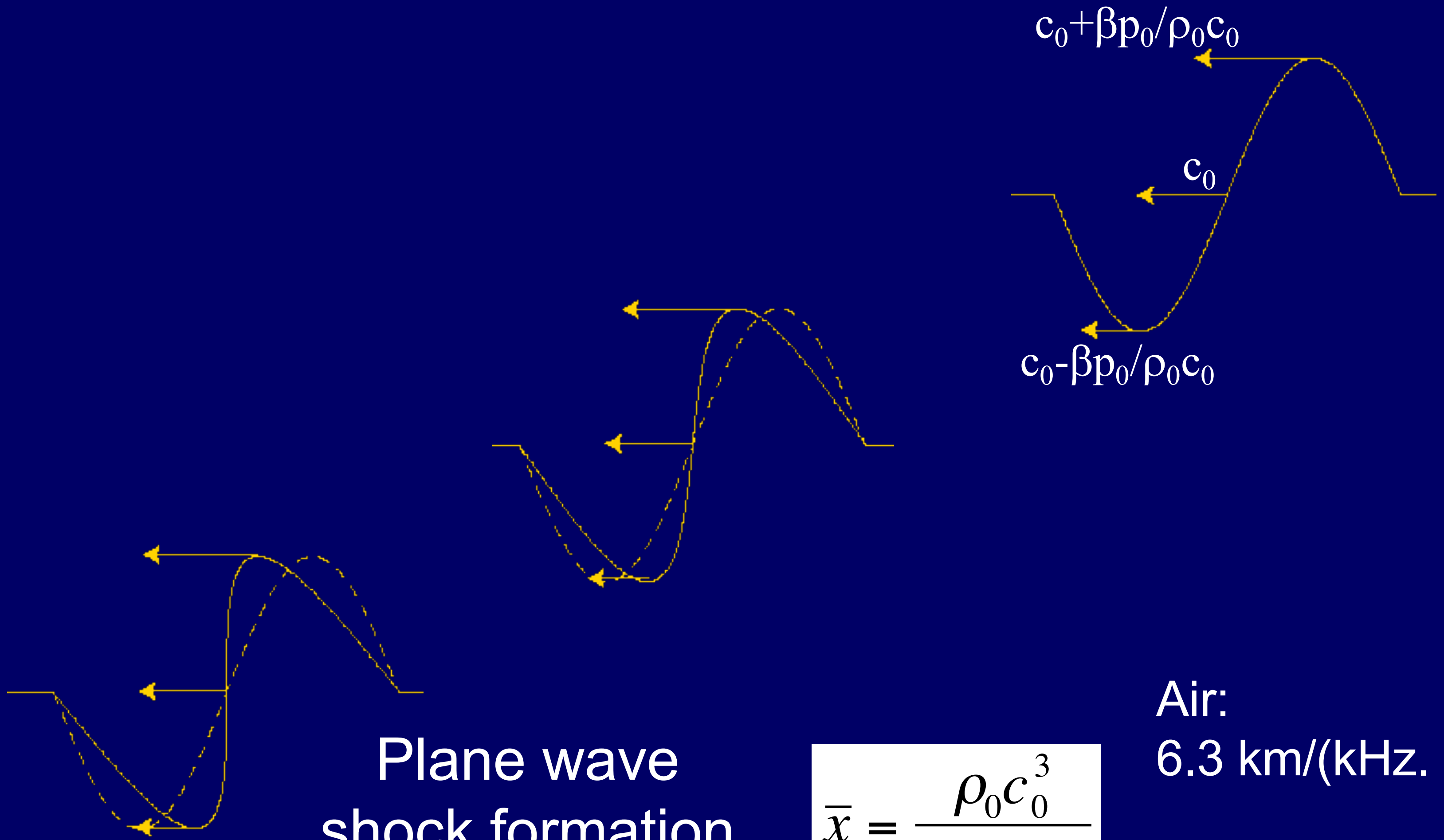
Sinusoidal source and Poisson solution

$$P = \sum_{n=1}^{\infty} B_n(x)\sin(n\omega\tau)$$

At x=0   B1=1 all other Bn = 0.

$$B_n(\sigma) = \frac{2}{n\sigma} J_n(n\sigma)$$

Nonlinearity and Shock Formation



Plane wave shock formation distance

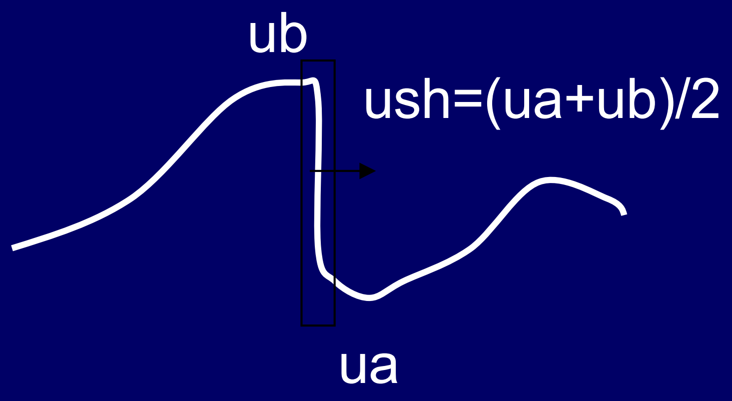
$$\bar{x} = \frac{\rho_0 c_0^3}{\beta p_0 2\pi f}$$

Air:
6.3 km/(kHz. Pa) 94 dB

Water:
100 mm/(MHz.MPa)

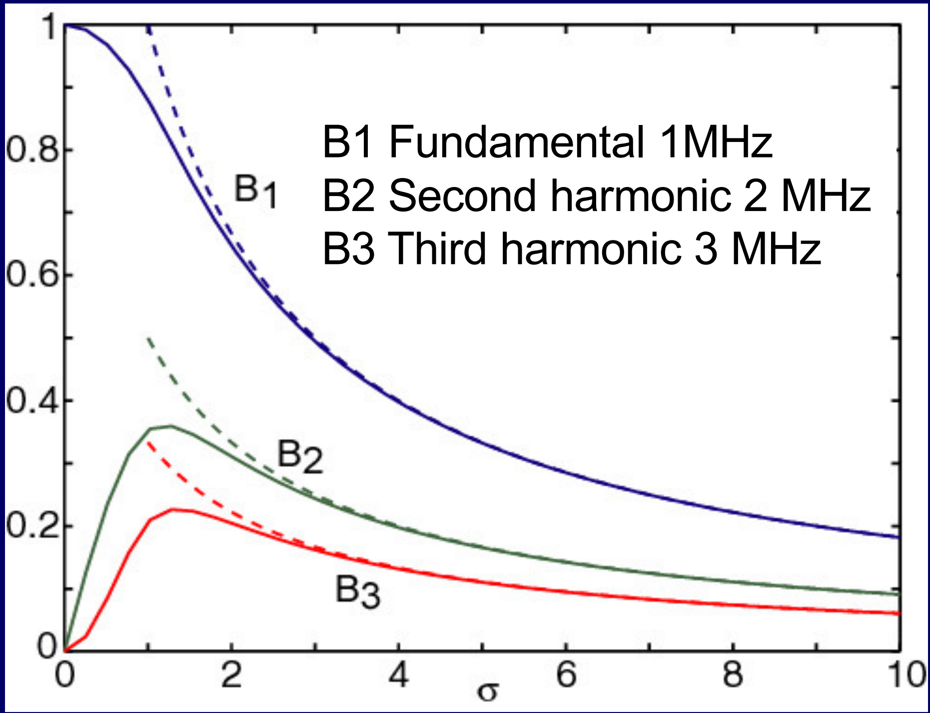
Weak Shock Theory

Smooth parts treated with **Poisson Solution**
Discontinuities treated with the **Shock Condition**



$$u_{sh} = c_0 + \beta \frac{p_a + p_b}{2\rho_0 c_0}$$

Harmonic Generation



Dimensionless distance

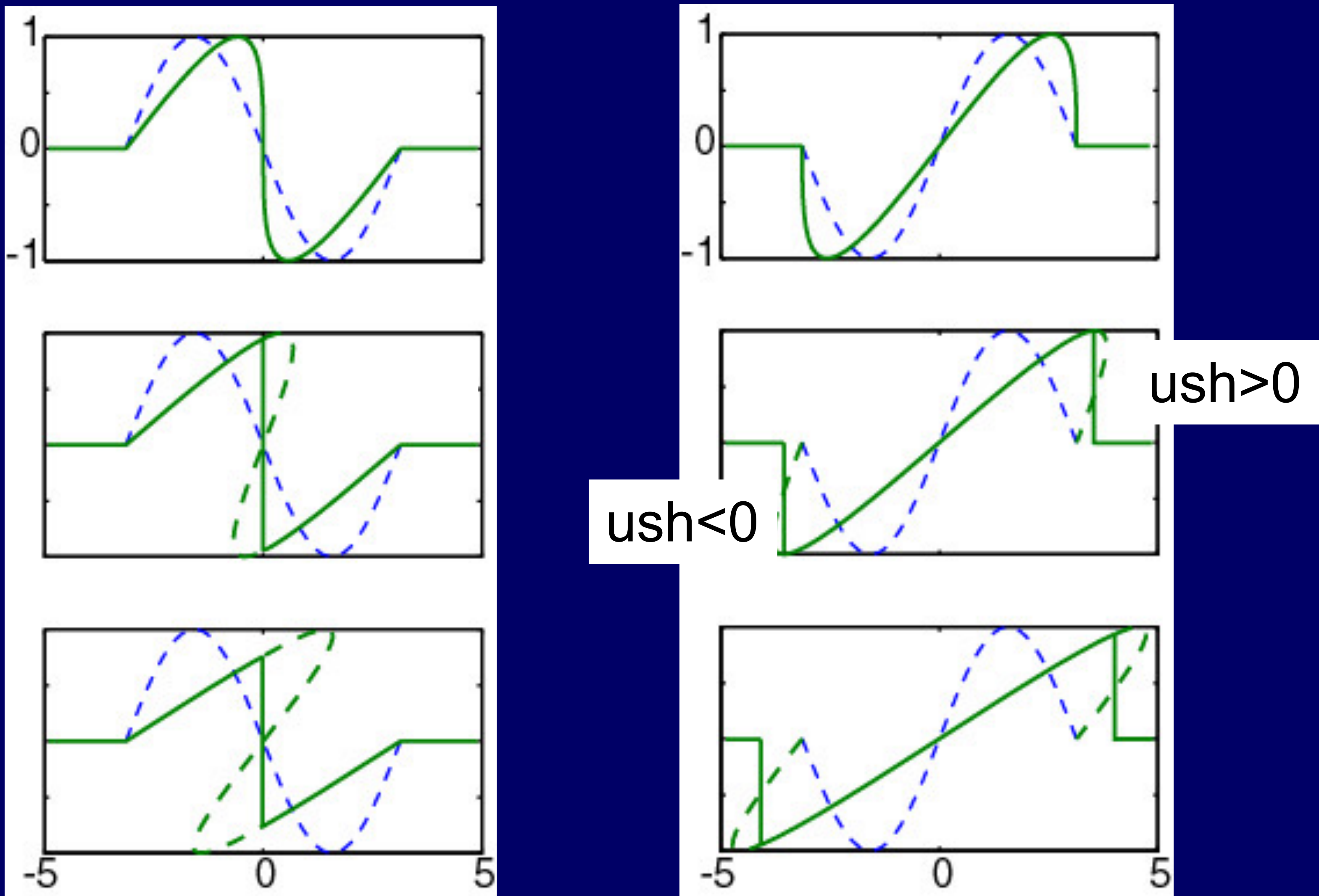
$$\sigma = \frac{x}{\bar{x}}$$

$\sigma << 1$ Neglect nonlinearity

$\sigma < 1$ Nonlinearity important

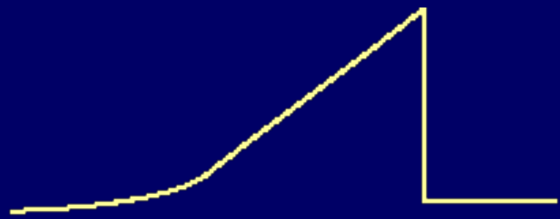
$\sigma > 1$ Shocks form

Pulses

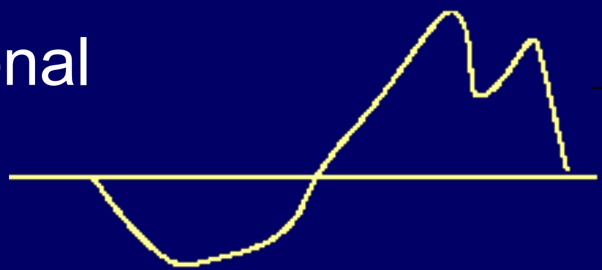


Other Transients

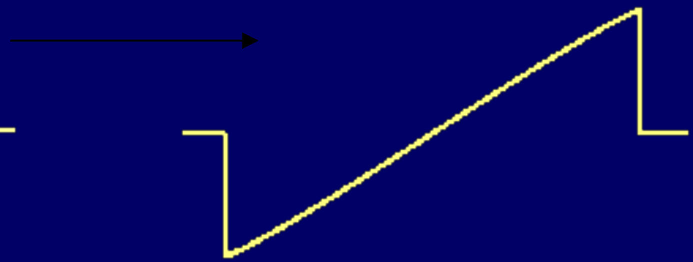
Explosive Sources



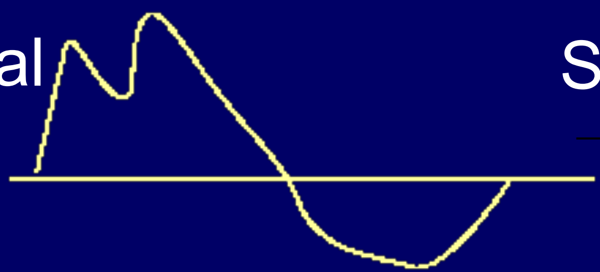
Leading Compressional Wave



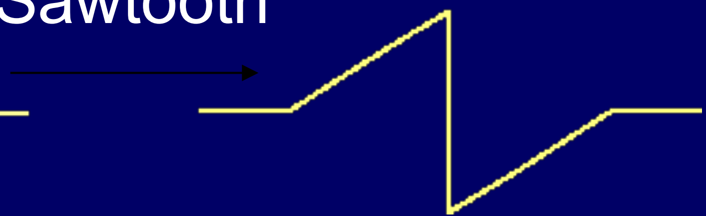
N-wave



Leading Rarefractional Wave



Sawtooth



Limitations of Weak Shock Theory

Shocks must be "thin" - that is appear as discontinuities.

Propagation distance must be SHORTER than the absorption length in medium

$$x < 1/\alpha$$

$$\bar{x} << 1/\alpha$$

$$1 << \frac{\beta\varepsilon k}{\alpha} = \Gamma$$

Gol'dberg number

Thermoviscous fluid:

Absorption coeffecient:

$$\alpha = \frac{\delta\omega_0^2}{2c_0^3}$$

δ diffusivity of sound; thermal conduction and viscosity

Burgers Equation

Include absorption but assume progressive plane waves:

$$\frac{\partial p}{\partial x} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau} + \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2}$$

Dimensionless:

$$\frac{\partial P}{\partial \sigma} = \frac{\partial P^2}{\partial \theta} + \frac{1}{\Gamma}\frac{\partial^2 P}{\partial \theta^2}$$

$$\sigma = \beta \varepsilon k x$$

$$\theta = \omega_0 \tau$$

$$\Gamma = \frac{\beta \varepsilon k}{\alpha}$$ Gol' dberg number

$$\alpha = \frac{\delta \omega_0^2}{2c_0^3}$$

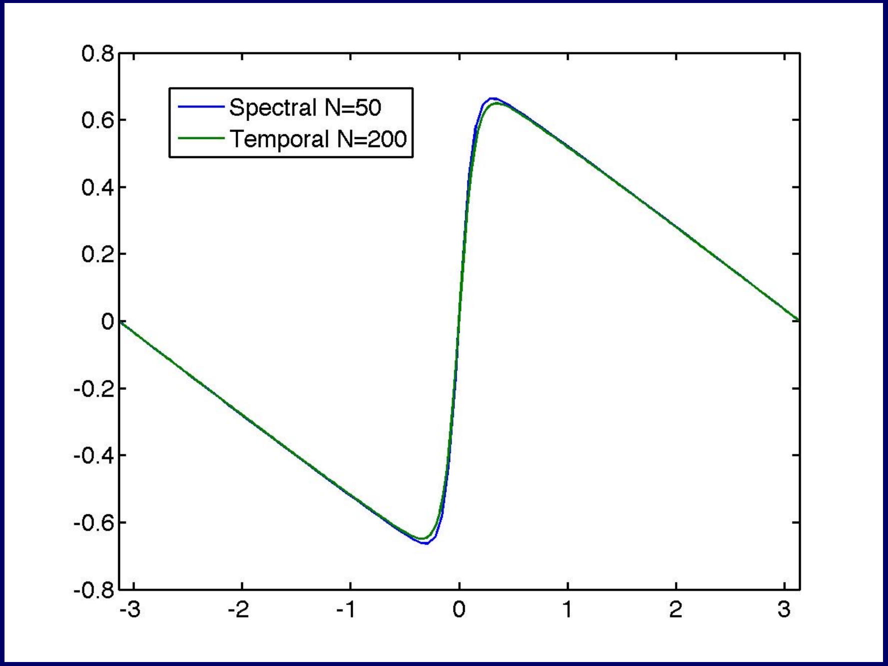
Hopf-Cole Transformation

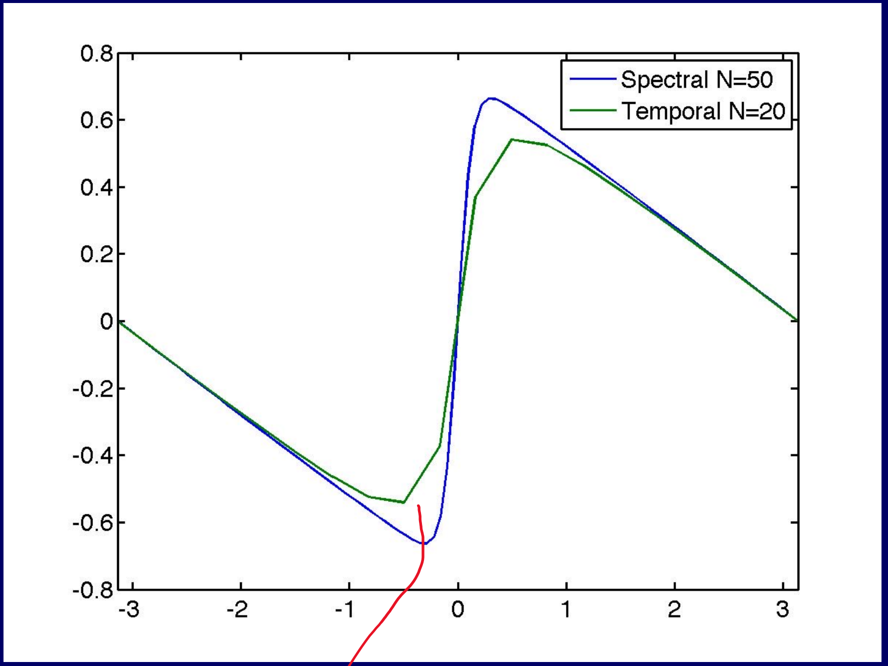
Hopf-Cole transformation yields an exact solution to Burgers equation. But still involves evaluating integrals.

Sinusoidal source wave produces an infinite sum over incomplete Bessel functions.

Burgers equation is more easily solved by numerical solutions in either the time or frequency domain.

Numerical: Time Domain Solution





Artificial absorption

Numerical: Frequency Domain

$$P(\sigma,\theta) = \sum_{m=1}^{M}\left(P_n(\sigma)e^{jn\theta} + P_n^*(\sigma)e^{-jn\theta}\right)/2$$

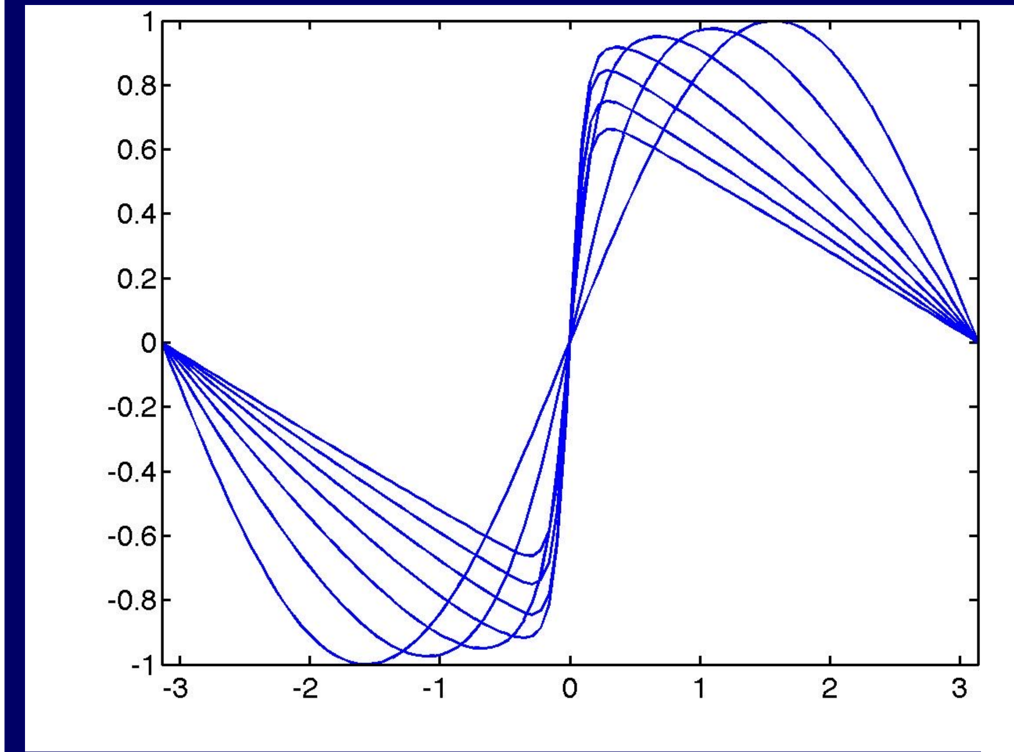
Coupled Ordinary Differential Equations

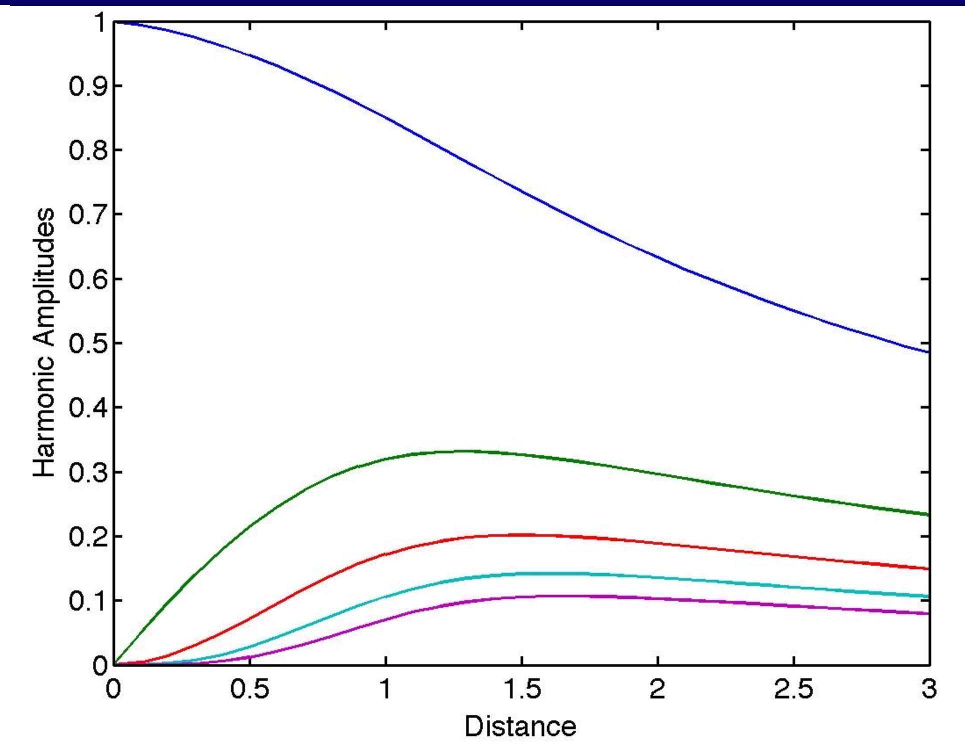
$$\frac{dP_n}{d\sigma} = -An^2P_n + j\frac{n}{4}\left(\sum_{m=1}^{n-1}P_mP_{n-m} + \sum_{m=n+1}^{M}P_mP_{m-n}^*\right)$$

Multiplication -> Fast ☺

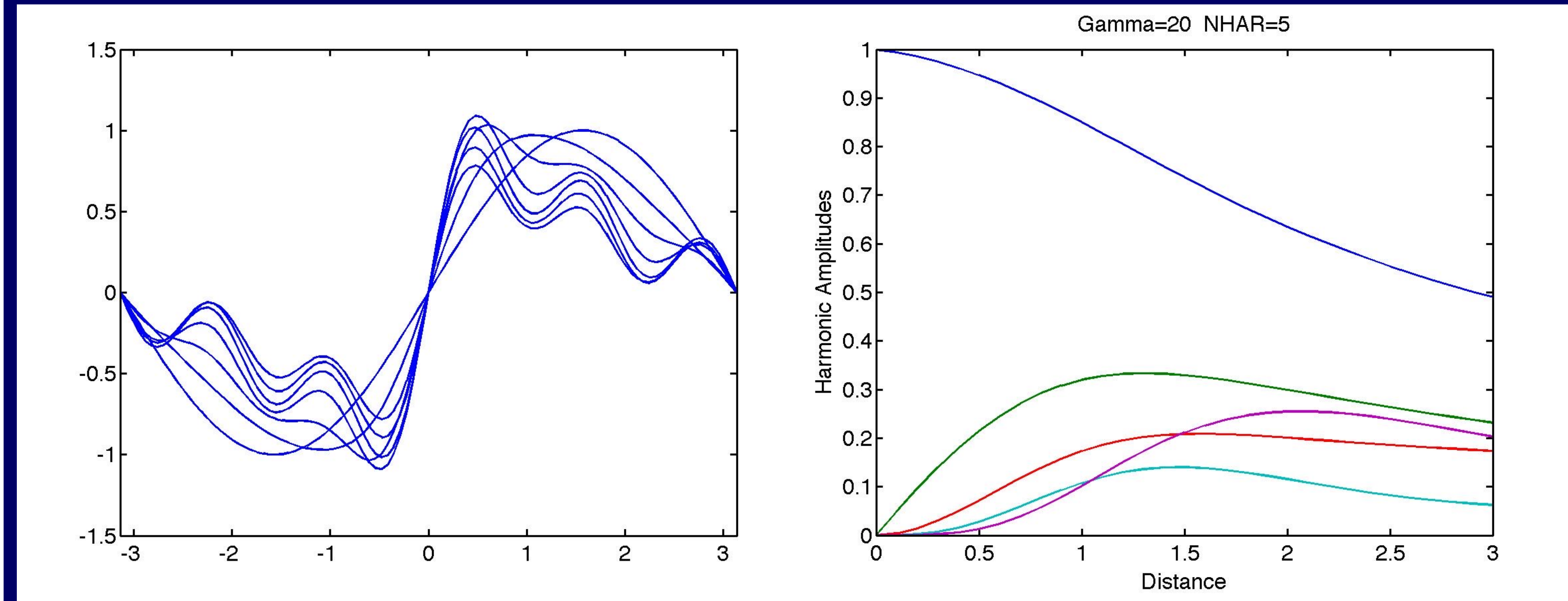
Convolution -> Slow ☹

Gamma=20 NHAR=50





Gamma=20 NHAR=5

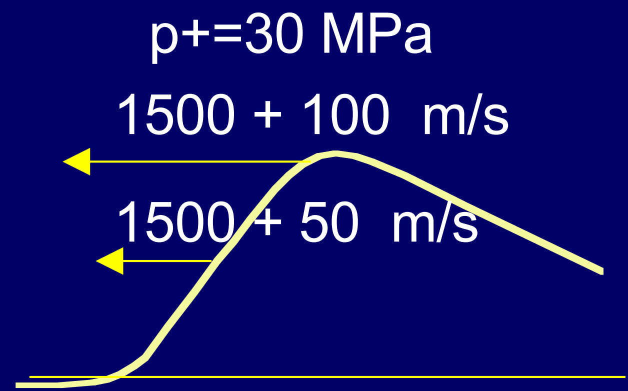


Harmonics not absorbed and reflected back

What is a Shock Wave?

Speed: $c = c_0 + \beta p/\rho_0 c_0$

β coefficient of nonlinearity (5)
ρ_0 density (1000 kg/m^3)
p acoustic pressure



Nonlinearity Steepens the Wave

Absorption Smooths the Wave

Progressive plane wave in retarded frame

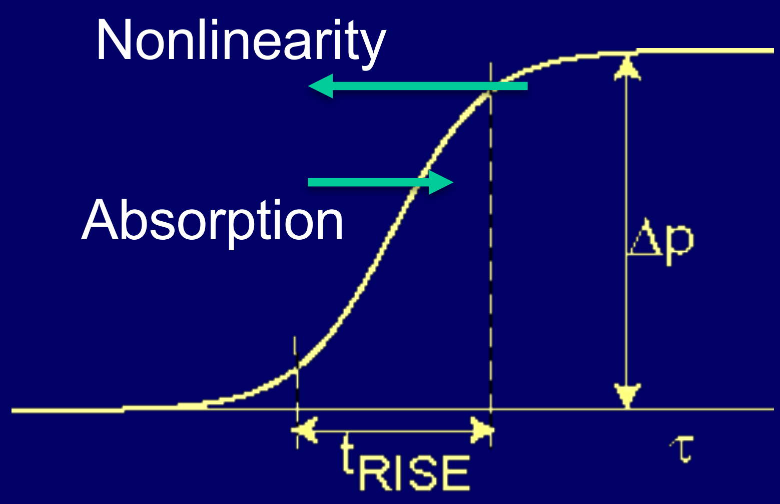
$$\frac{\partial p}{\partial x} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau} + \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2}$$

Taylor Shock

Stationary solution

$$p = \frac{\Delta p}{2}\left(1 + \tanh\left(\frac{\beta \Delta p}{2\rho_0 \delta}\tau\right)\right)$$

δ diffusivity of sound
(thermoviscous absorption)



Taylor shock thickness:

$$t_{RISE} = \frac{4\rho_0 \delta}{\beta \Delta p}$$

In water: $t_{RISE}\Delta p$=0.005 Pa.s
=5 ns.MPa

Arbitrary Attenuation

Frequency domain

$$\frac{\partial P(\omega)}{\partial x} = -\alpha(\omega)P(\omega)$$

$$\frac{dP_n}{d\sigma} = -A_n P_n + j\frac{n}{4}\left(\sum_{m=1}^{n-1} P_m P_{n-m} + \sum_{m=n+1}^{M} P_m P_{m-n}^{*}\right)$$

But A_n MUST be causal. The real and imaginary parts (attenuation and dispersion) must satisfy the Kramers-Kronig relations. For example for power law: f^y

$$A_n = A_1 n^y \left(1 + j\left(1 - n^{1-y}\right)\tan(\pi y / 2)\right)$$

Wallace et al JASA 1991

Power Law Attenuation and Dispersion

Kramers-Kronig Relations

$$\bar{\alpha} = \alpha_0 \omega^n$$

Szabo, J. Acoust Soc. Am. 97:14 (1995)
Waters et al JASA 108:556 (2000).

$$\frac{1}{c(\omega)} = \frac{1}{c(\omega_0)} + \alpha_0 \tan(n\pi/2)\left(\omega^{n-1} - \omega_0^{n-1}\right)$$

n=1

$$\frac{1}{c(\omega)} = \frac{1}{c(\omega_0)} - \frac{2}{\pi}\alpha_0 \ln\left(\frac{\omega}{\omega_0}\right)$$

n=2

No dispersion

Attenuation - Time Domain

Frequency domain　　　　Time domain

$$\frac{\partial P(\omega)}{\partial x} = -\alpha(\omega)P(\omega) \Leftrightarrow \frac{\partial p(\tau)}{\partial x} = h(\tau) * p(\tau)$$

Convolution
-> Slow ☹

Attenuation via Relaxation Processes

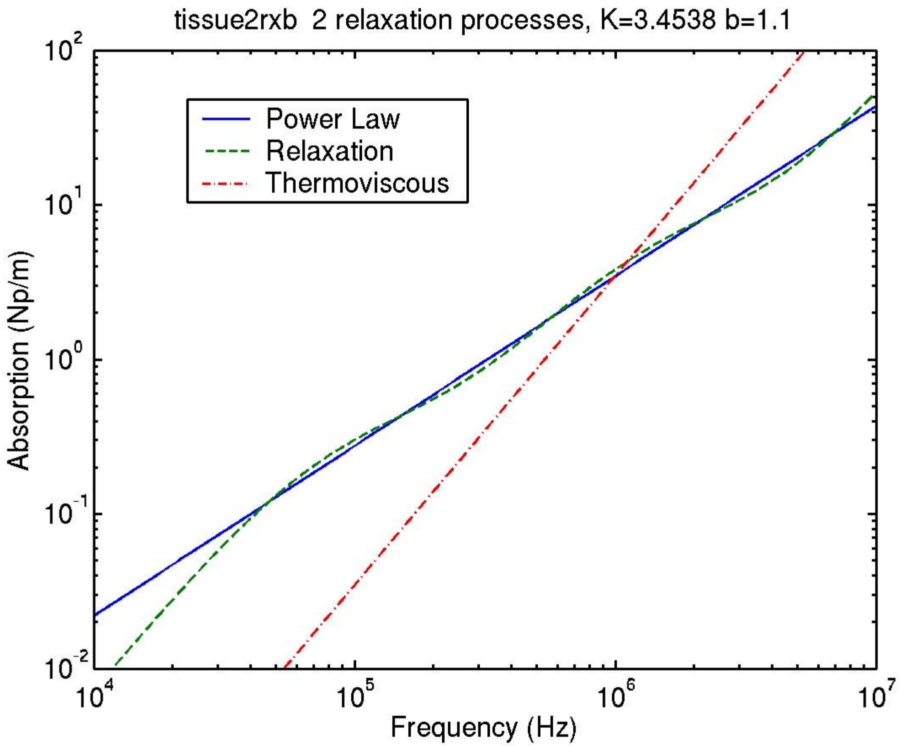
$$\frac{\partial p}{\partial z} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau} + \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2} + \sum_{\nu} \frac{c'_{\nu}}{c_0^2}\int_{-\infty}^{\tau} \frac{\partial^2 p}{\partial t''^2} e^{-(\tau - t'')/t_{\nu}} dt''$$

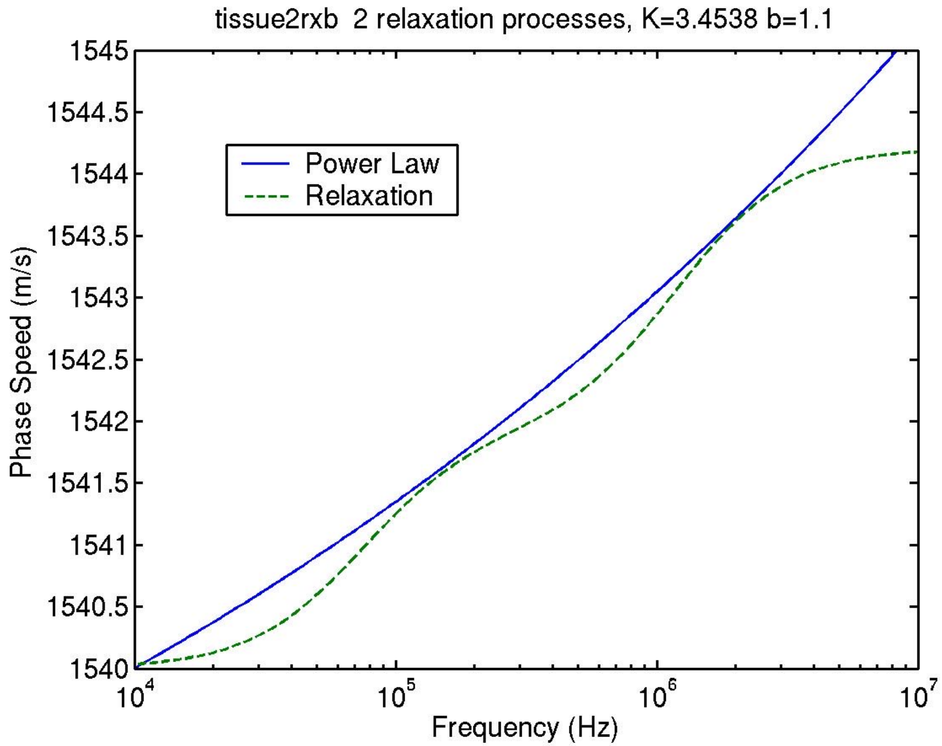
Thermoviscous Absorption

Relaxation Process

t_{ν} relaxation time

c'_{ν} dispersion

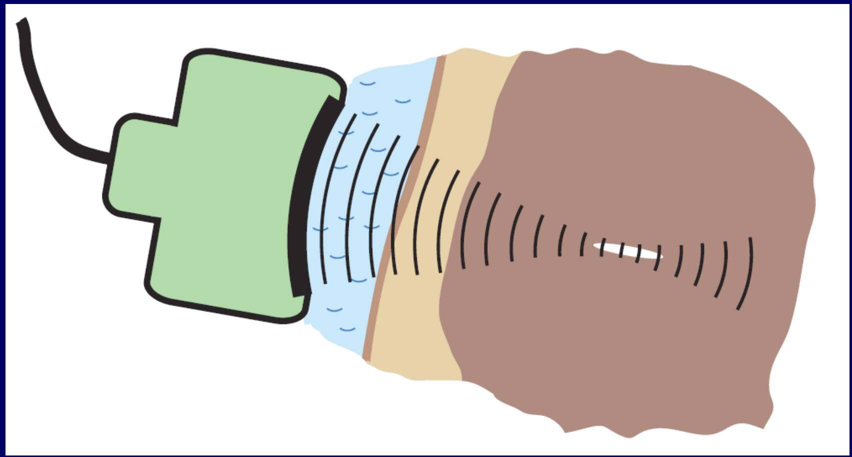




Example for Biomedical Ultrasound

Diffraction

Focal lengths $\boxed{10\text{-}150 \text{ mm}}$



Absorption

Soft tissue 0.3 dB/cm/MHz

At 1 MHz length scale $\boxed{30 \text{ mm}}$

Nonlinearity

Plane wave shock formation distance in tissue

$$\bar{x} \approx \frac{100}{p_0 f} \ \mathrm{mm} \cdot \mathrm{MPa} \cdot \mathrm{MHz}$$

At 1 MHz and 3 MPa length scale $\boxed{30 \text{ mm}}$

Self-consistent absorption model

- Models for attenuation are empirical and account for both absorption and scattering
- Nonlinear harmonics have shorter wavelengths - therefore length scale for sub-wavelength scatters decreases
  - 1500 μm wavelength 10 μm is small
  - 30 μm wavelength 10 μm is not small

Westervelt Equation

1 MHz: wavelength of 1500 μm
15 harmonics: wavelength of 100 μm
Volume = 150 mm x 80 mm x 80 mm
Propagation x Lateral Dimensions

Finite-difference 5 μm voxel size
•40,000 x 16,000 x 16,000 = 10,000 billion voxels
•Remember 4 time steps
•40,000 billion numbers = 160,000 GB (single float)
•Time step 3 ns

•150 mm needs 30,000 steps:
•5 point stencil 12 FLOPS/step: 58 000 petaFLOP

Supercomputers
Frontier 1102 petaFLOPS (June 2022)
Summit 122 petaFLOPS (June 2018)

3 harmonics
4 points per wavelength
Memory reduction:
25^3 = 16,000

10s GB

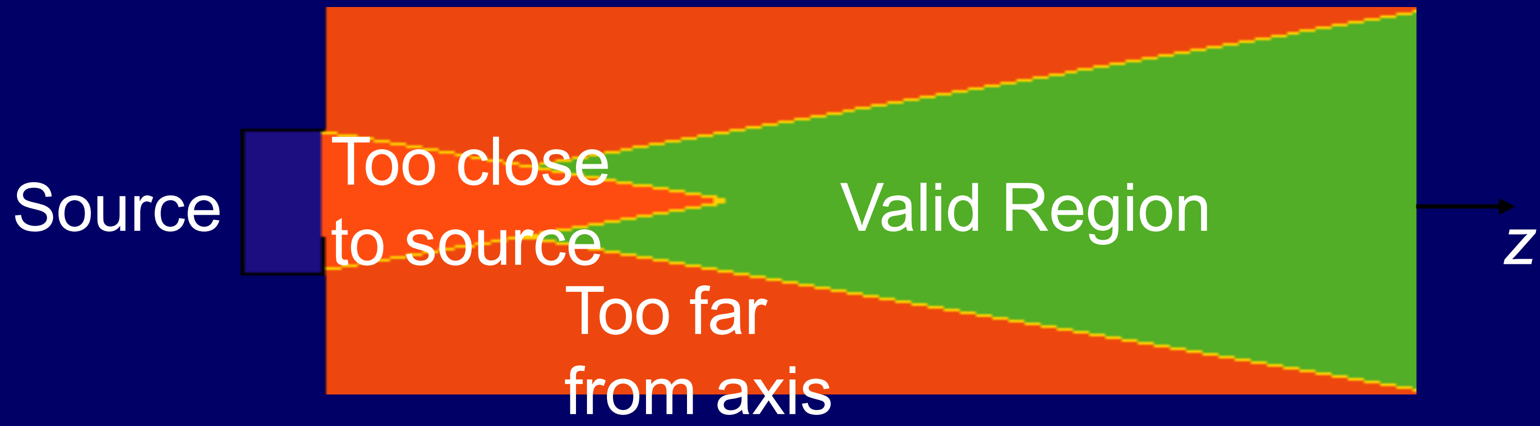
150 teraFLOPs

Treeby and Cox
UCL

KZK Equation

Zabolotskaya and Khokhlov, Sov. Phys. Acoust. 1969
Kuznetsov, Sov. Phys. Acoust. 1971.

Diffraction in Parabolic Approximation: angles < 20°



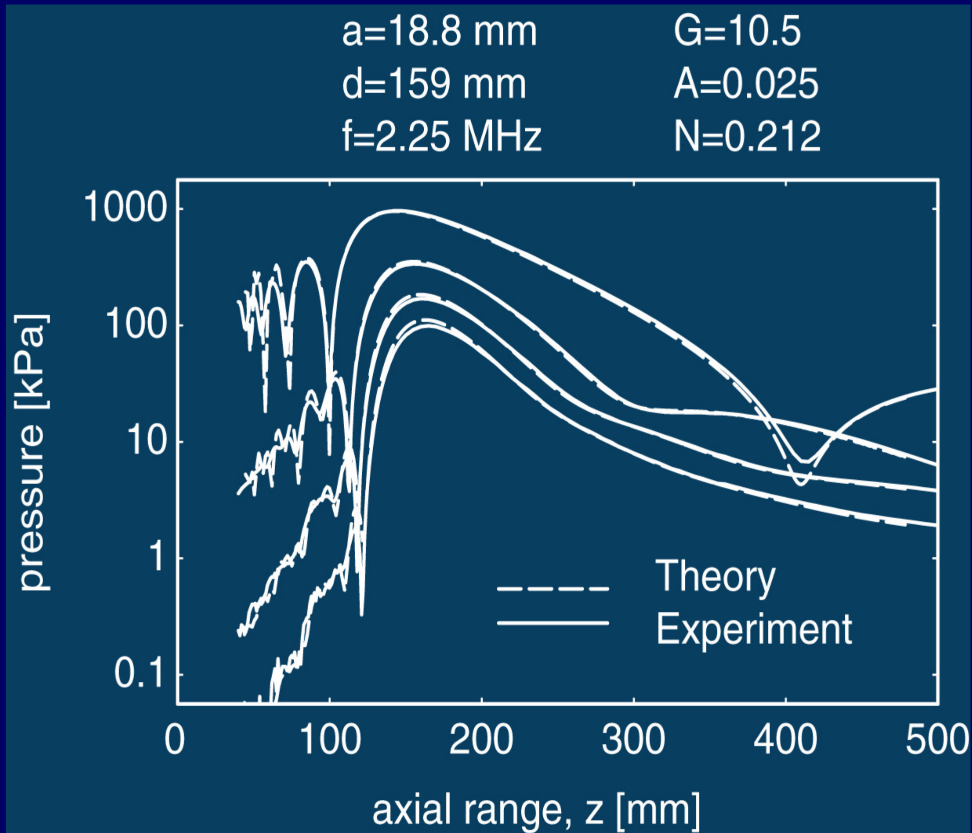
$$\frac{\partial p}{\partial z} = \frac{c_0}{2}\int_{-\infty}^{\tau} \nabla_\perp^2 p\, d\tau + \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau} + \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2}$$

Diffraction Nonlinearity Absorption

Retarded time $\tau = t - z/c_0$

Nonlinear propagation in water

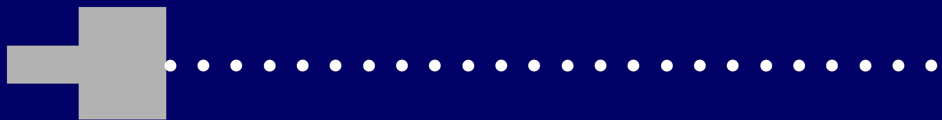
KZK and experiment



Circular x-ducer
Focused , CW

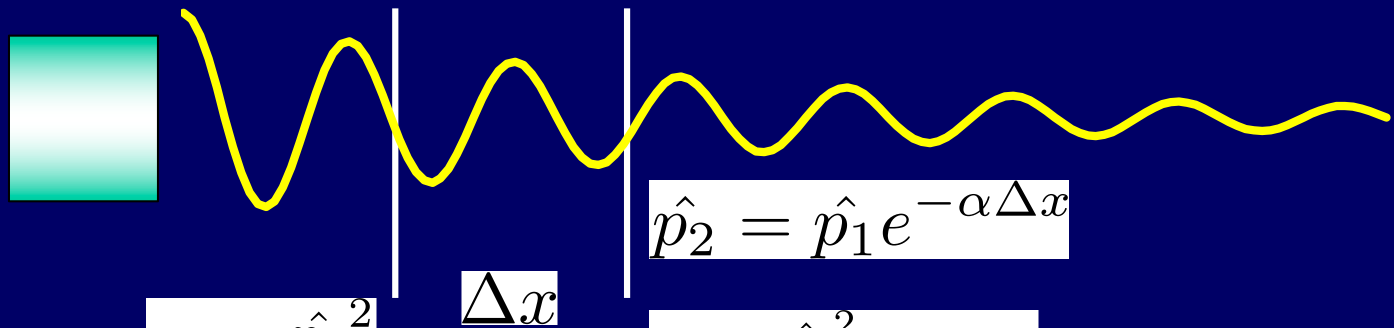
Axial pressure for 4 harmonic components

Fund
2nd hrm
3rd hrm
4th hrm



Averkiou and Hamilton, JASA 1995

Power Lost due to Attenuation



$$\hat{p}_2 = \hat{p}_1 e^{-\alpha \Delta x}$$

$$I_1 = \frac{\hat{p}_1^{\,2}}{2Z}$$

$$I_2 = \frac{\hat{p}_1^{\,2}}{2Z} e^{-2\alpha \Delta x}$$

$$\frac{I_1 - I_2}{\Delta x} = 2\alpha I$$ W/m^3 Power/Volume

In general

$$q_s = -\nabla I$$ W/m^3 Heat transfer rate per unit volume

Pennes Bioheat Transfer Equation (BHTE)

* H. H. Pennes, *Journal of Applied Physiology*, 2: 93-122, 1948

BUBBL

$$\rho_t C_t \frac{\partial T}{\partial t} = K_t \nabla^2 T - w_b C_b (T - T_\infty) + q_s$$

$$q_s = -\nabla I$$

Conduction Perfusion

Where c_T is the specific heat of the tissue ~4000 J/kg/K

Neglected: conduction and convection (perfusion by blood).

$$\frac{\Delta T}{\Delta t} = \frac{\alpha \hat{p}^2}{\rho^2 c_v c}$$

At 1 MHz and 1MPa

~1 ^{o}C/s

Recap

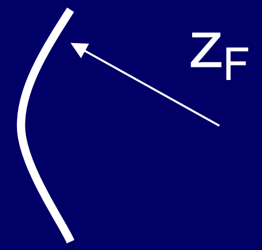
- Fluid equations: $p_a \ll \rho_0 c_0^2$
- Nonlinear effects are cumulative
  Shock formation distance: $\bar{x} = \dfrac{\rho_0 c_0^3}{\beta p_0 2\pi f}$
- Shock rise time: $t_{RISE} = \dfrac{4\rho_0\delta}{\beta\Delta p}$
- Model equations:
  - Burgers
  - KZK
  - Westervelt

Artifacts:
Time: smoothing of shocks
Frequency: harmonic reflections

- Full modelling of diffraction, nonlinearity, absorption and heterogeneity is numerically challenging

Focused Source

Length scales: Focal length
Nonlinear
Absorption
Diffraction



$$\sigma = z / z_F$$
$$P = p / p_0$$
$$R = r / a$$
$$\theta = \omega\tau$$

$$G = z_R / z_F$$

Phase at source

$$P_s = F(R, \omega\tau + GR^2)$$

$$\frac{\partial P}{\partial \sigma} = \frac{1}{4G}\int_{-\infty}^{\theta}\left(\frac{\partial^2 P}{\partial R^2} + \frac{1}{R}\frac{\partial P}{\partial R}\right)d\theta' + NP\frac{\partial P}{\partial \theta} + A\frac{\partial^2 P}{\partial \theta^2}$$

Literature

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