UNIVERSITY OF OXFORD

Short Course on Nonlinear Acoustics

22nd International Symposium on Nonlinear Acoustics

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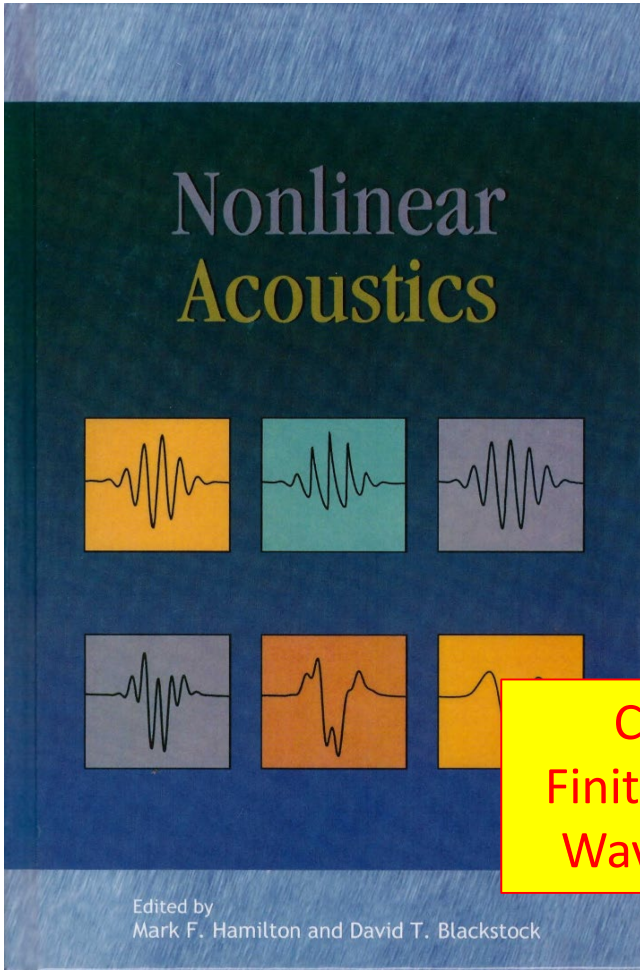
Part III.
Nonlinear Elastic Waves

Useful reference (mainly for fluids)



Chapter 9:
Finite-Amplitude
Waves in Solids

This book will be republished soon by Springer with permanent open access, downloadable for free in its entirety

Chapter authors (15 chapters)

- David. R. Bacon
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- Taylor G. Wang
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Longitudinal and Shear Waves

Landau theory for isotropic solids

Finally, we may briefly discuss how we can set up the equations of motion, allowing for the anharmonic terms. The strain tensor must now be given by the complete expression (1.3):

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right), \qquad (26.1)$$

in which the terms quadratic in u_i can not be neglected. Next, the general expression for the energy density† $\mathscr{E}$, in bodies having a given symmetry, must be written as a scalar formed from the components of the tensor u_{ik} and some constant tensors characteristic of the substance involved; this scalar will contain terms up to a given power of u_{ik}. Substituting the expression (26.1) for u_{ik} and omitting terms in u_i of higher orders than that power, we find the energy $\mathscr{E}$ as a function of the derivatives $\partial u_i/\partial x_k$ to the required accuracy.

In order to obtain the equations of motion, we notice the following result. The variation $\delta\mathscr{E}$ may be written

$$\delta\mathscr{E} = \frac{\partial\mathscr{E}}{\partial(\partial u_i/\partial x_k)}\delta\frac{\partial u_i}{\partial x_k},$$

or, putting

$$\sigma_{ik} = \frac{\partial\mathscr{E}}{\partial(\partial u_i/\partial x_k)}, \qquad (26.2)$$

$$\delta\mathscr{E} = \sigma_{ik}\frac{\partial\delta u_i}{\partial x_k} = \frac{\partial}{\partial x_k}(\sigma_{ik}\delta u_i) - \delta u_i\frac{\partial\sigma_{ik}}{\partial x_k}.$$

The coefficients of $-\delta u_i$ are the components of the force per unit volume of the body. They formally appear the same as before, and so the equations of motion can again be written

$$\rho_0\ddot{u}_i = \partial\sigma_{ik}/\partial x_k, \qquad (26.3)$$

where ρ_0 is the density of the undeformed body, and the components of the tensor σ_{ik} are now given by (26.2), with $\mathscr{E}$ correct to the required accuracy. The tensor σ_{ik} is no longer symmetrical.

It should be emphasized that σ_{ik} is no longer the momentum flux density (the stress tensor). In the ordinary theory this interpretation was derived by integrating the body force density $\partial\sigma_{ik}/\partial x_k$ over the volume of the body. This derivation depended on the fact that, in performing the integration, we made no distinction between the coordinates of points in the body before and after the deformation. In subsequent approximations, however, this distinction must be made, and the surface bounding the region of integration is not the same as the actual surface of the region considered after the deformation.

It has been shown in §2 that the symmetry of the tensor σ_{ik} is due to the conservation of angular momentum. This result no longer holds, since the angular momentum density is not $x_i\dot{u}_k - x_k\dot{u}_i$ but $(x_i+u_i)\dot{u}_k - (x_k+u_k)\dot{u}_i$.

PROBLEM

Write down the general expression for the elastic energy of an isotropic body in the third approximation.

SOLUTION. From the components of a symmetrical tensor of rank two we can form two quadratic scalars (u_{ik}^2 and u_{ll}^2) and three cubic scalars (u_{ll}^3, $u_{ll}u_{ik}^2$ and $u_{ik}u_{il}u_{kl}$). Hence the most general scalar containing terms

† We here use the internal energy $\mathscr{E}$, and not the free energy F, since adiabatic vibrations are involved.

quadratic and cubic in u_{ik}, with scalar coefficients (since the body is isotropic), is

$$\mathscr{E} = \mu u_{ik}^2 + (\tfrac{1}{2}K - \tfrac{1}{3}\mu)u_{ll}^2 + \tfrac{1}{3}A u_{ik}u_{il}u_{kl} + B u_{ik}^2 u_{ll} + \tfrac{1}{3}C u_{ll}^3;$$

the coefficients of u_{ik}^2 and u_{ll}^2 have been expressed in terms of the moduli of compression and rigidity, and A, B, C are three new constants. Substituting the expression (26.1) for u_{ik} and retaining terms up to and including the third order, we find the elastic energy to be

$$\begin{aligned}\mathscr{E} = {} & \tfrac{1}{4}\mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)^2 + (\tfrac{1}{2}K - \tfrac{1}{3}\mu)\left(\frac{\partial u_l}{\partial x_l}\right)^2 + \\ & + (\mu + \tfrac{1}{4}A)\frac{\partial u_i}{\partial x_k}\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k} + (\tfrac{1}{2}B + \tfrac{1}{2}K - \tfrac{1}{3}\mu)\frac{\partial u_i}{\partial x_i}\left(\frac{\partial u_l}{\partial x_k}\right)^2 + \\ & + \tfrac{1}{12}A\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_l}\frac{\partial u_l}{\partial x_i} + \tfrac{1}{2}B\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_l} + \tfrac{1}{3}C\left(\frac{\partial u_l}{\partial x_l}\right)^3.\end{aligned}$$

Five elastic constants describe nonlinear propagation of (quasi-plane) compressional waves,

- two "second order" elastic constants (shear modulus μ and bulk modulus K), and
- three "third order" elastic constants (A, B and C).
- Coefficient of nonlinearity for plane compressional (longitudinal) waves:

$$\beta_l = -\left(\frac{3}{2} + \frac{A + 3B + C}{\mu + \frac{4}{3}K}\right)$$

Landau and Lifshitz, *Theory of Elasticity*

Collinear wave interaction with quadratic nonlinearity in an isotropic solid (Gol'dberg 1961)

For particle displacement field

$$\mathbf{u}(x,\ t) = \mathbf{i}u_x(x,\ t) + \mathbf{j}u_y(x,\ t) + \mathbf{k}u_z(x,\ t),$$

the equations for plane wave propagation of u_x, u_y, and u_z components in the x direction are

$$\rho_0 \frac{\partial^2 u_x}{\partial t^2} - \alpha \frac{\partial^2 u_x}{\partial x^2} = \beta \frac{\partial^2 u_x}{\partial x^2} \cdot \frac{\partial u_x}{\partial x} + \gamma \left(\frac{\partial^2 u_y}{\partial x^2} \cdot \frac{\partial u_y}{\partial x} + \frac{\partial^2 u_z}{\partial x^2} \cdot \frac{\partial u_z}{\partial x} \right),$$

A plane shear wave (u_y) can generate, via quadratic nonlinearity, a longitudinal wave (u_x)

$$\rho_0 \frac{\partial^2 u_z}{\partial t^2} - \mu \frac{\partial^2 u_z}{\partial x^2} = \gamma \left(\frac{\partial^2 u_z}{\partial x^2} \cdot \frac{\partial u_x}{\partial x} + \frac{\partial^2 u_x}{\partial x^2} \cdot \frac{\partial u_z}{\partial x} \right),$$

where

$$\alpha = K + \frac{4}{3}\mu, \quad \beta = 3\alpha + 2A + 6B + 2C, \quad \gamma = \alpha + \frac{A}{2} + B.$$

Collinear wave interaction with quadratic nonlinearity in an isotropic solid (Gol'dberg 1961)

For particle displacement field

$$\mathbf{u}(x,\ t) = \mathbf{i}u_x(x,\ t) + \mathbf{j}u_y(x,\ t) + \mathbf{k}u_z(x,\ t),$$

the equations for plane wave propagation of u_x, u_y, and u_z components in the x direction are

$$\rho_0 \frac{\partial^2 u_x}{\partial t^2} - \alpha \frac{\partial^2 u_x}{\partial x^2} = \beta \frac{\partial^2 u_x}{\partial x^2} \cdot \frac{\partial u_x}{\partial x} + \gamma \left(\frac{\partial^2 u_y}{\partial x^2} \cdot \frac{\partial u_y}{\partial x} + \frac{\partial^2 u_z}{\partial x^2} \cdot \frac{\partial u_z}{\partial x} \right),$$

$$\rho_0 \frac{\partial^2 u_y}{\partial t^2} - \mu \frac{\partial^2 u_y}{\partial x^2} = \gamma \left(\frac{\partial^2 u_y}{\partial x^2} \cdot \frac{\partial u_x}{\partial x} + \frac{\partial^2 u_x}{\partial x^2} \cdot \frac{\partial u_y}{\partial x} \right),$$

A plane longitudinal wave (u_x) cannot by itself generate a shear wave (u_y)

where

$$\alpha = K + \frac{4}{3}\mu, \quad \beta = 3\alpha + 2A + 6B + 2C, \quad \gamma = \alpha + \frac{A}{2} + B.$$

Collinear wave interaction with quadratic nonlinearity in an isotropic solid (Gol'dberg 1961)

For particle displacement field

$$\mathbf{u}(x,\ t) = \mathbf{i}u_x(x,\ t) + \mathbf{j}u_y(x,\ t) + \mathbf{k}u_z(x,\ t),$$

the equations for plane wave propagation of u_x, u_y, and u_z components in the x direction are

$$\rho_0 \frac{\partial^2 u_x}{\partial t^2} - \alpha \frac{\partial^2 u_x}{\partial x^2} = \beta \frac{\partial^2 u_x}{\partial x^2} \cdot \frac{\partial u_x}{\partial x} + \gamma \left(\frac{\partial^2 u_y}{\partial x^2} \cdot \frac{\partial u_y}{\partial x} + \frac{\partial^2 u_z}{\partial x^2} \cdot \frac{\partial u_z}{\partial x} \right),$$

$$\rho_0 \frac{\partial^2 u_y}{\partial t^2} - \mu \frac{\partial^2 u_y}{\partial x^2} = \gamma \left(\frac{\partial^2 u_y}{\partial x^2} \cdot \frac{\partial u_x}{\partial x} + \frac{\partial^2 u_x}{\partial x^2} \cdot \frac{\partial u_y}{\partial x} \right),$$

Finally, a plane shear wave (u_y) cannot, by itself, generate another shear wave (u_y), for example a second harmonic, through quadratic nonlinearity

where

$$\alpha = K + \frac{4}{3}\mu, \quad \beta = 3\alpha + 2A + 6B + 2C, \quad \gamma = \alpha + \frac{A}{2} + B.$$

Alternative notation for third-order elastic constants

RELATIONS BETWEEN THIRD-ORDER ELASTIC CONSTANTS FOR ISOTROPIC SOLIDS

Toupin & Bernstein (1961)	Murnaghan (1951)	Bland (1969)	Eringen & Suhubi (1974)	Standard, c_{IJK}	
$\nu_1 = 2\mathcal{C}$	$l = \mathcal{B} + \mathcal{C}$	$\alpha = \frac{1}{3}\mathcal{C}$	$l_E = \frac{1}{3}\mathcal{A} + \mathcal{B} + \frac{1}{3}\mathcal{C}$	$c_{123} = 2\mathcal{C}$	$c_{111} = 2\mathcal{A} + 6\mathcal{B} + 2\mathcal{C}$
$\nu_2 = \mathcal{B}$	$m = \frac{1}{2}\mathcal{A} + \mathcal{B}$	$\beta = \mathcal{B}$	$m_E = -\mathcal{A} - 2\mathcal{B}$	$c_{144} = \mathcal{B}$	$c_{112} = 2\mathcal{B} + 2\mathcal{C}$
$\nu_3 = \frac{1}{4}\mathcal{A}$	$n = \mathcal{A}$	$\gamma = \frac{1}{3}\mathcal{A}$	$n_E = \mathcal{A}$	$c_{456} = \frac{1}{4}\mathcal{A}$	$c_{166} = \frac{1}{2}\mathcal{A} + \mathcal{B}$

Norris, Chap. 9, *Nonlinear Acoustics* (2008)

Burgers and KZK equations for compressional waves

Burgers equation (Mendousse 1953, Lighthill 1956) for compressional waves:

$$\frac{\partial v_z}{\partial z} = \frac{\beta_l}{c_l^2} v_z \frac{\partial v_z}{\partial \tau} + \frac{\delta}{2c_l^3} \frac{\partial^2 v_z}{\partial \tau^2} \qquad v_z = \frac{\partial u_z}{\partial \tau} \qquad \tau = t - z/c_l$$

KZK equation for quasi-plane compressional waves in diffracting beams (Zabolotskaya, Sov. Phys. Acoust. 1986):

$$\frac{\partial v_z}{\partial z} = \frac{\beta_l}{c_l^2} v_z \frac{\partial v_z}{\partial \tau} + \frac{\delta}{2c_l^3} \frac{\partial^2 v_z}{\partial \tau^2} + \frac{c_l}{2} \int_{-\infty}^{\tau} (\nabla_\perp^2 v_z)\, d\tau'$$

Coefficient of nonlinearity:

$$\beta_l = -\left(\frac{3}{2} + \frac{A + 3B + C}{\mu + \frac{4}{3}K} \right)$$

- Whereas the coefficient of nonlinearity is positive for fluids, <u>it can be negative for some solids</u>, such as fused quartz

Numerical solution in frequency domain as for fluids

Dimensionless form:

$$\frac{\partial V_x}{\partial \sigma} = V_x \frac{\partial V_x}{\partial \theta} + \frac{1}{\Gamma} \frac{\partial^2 V_x}{\partial \theta^2}$$

Expand V_x in Fourier series:

$$V_x(\sigma, \theta) = \frac{1}{2} \sum_{n=1}^{N} V_n(\sigma) e^{jn\theta} + \text{c.c.}$$

N coupled spectral equations obtained:

$$\frac{dV_n}{d\sigma} = j\frac{n}{4} \left(\underbrace{\sum_{m=1}^{n-1} V_m V_{n-m}}_{\text{sum frequency generation}} + 2 \underbrace{\sum_{m=n+1}^{N} V_m V^*_{m-n}}_{\text{difference frequency generation}} \right) - \frac{n^2}{\Gamma} V_n \qquad n = 1, 2, \cdots N$$

- Easily solved by Runge-Kutta
- Arbitrary absorption and dispersion taken into account by replacing coefficient n^2/Γ with $(A_n + jD_n)$.

A cautionary word about *ad hoc* power law attenuation

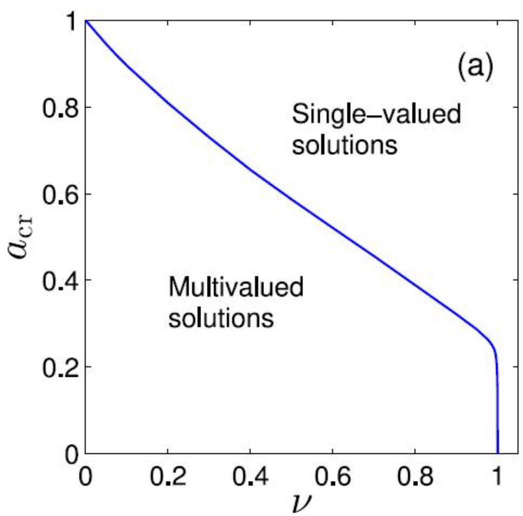
Arbitrary attenuation and dispersion introduced:

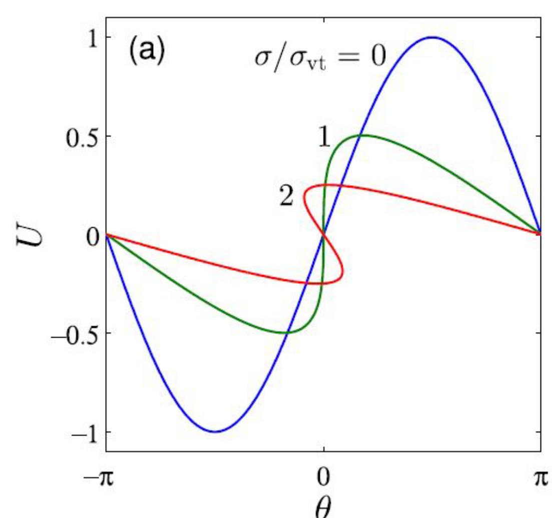
$$\frac{dV_n}{d\sigma} = j\frac{n}{4}\left(\sum_{m=1}^{n-1} V_m V_{n-m} + 2\sum_{m=n+1}^{N} V_m V_{m-n}^*\right) - (A_n + jD_n)V_n$$

Power law attenuation that satisfies Kramers-Kronig relation:

$$\alpha(\omega) = \delta_\nu \omega^\nu \cos\left(\frac{\nu\pi}{2}\right), \quad \frac{1}{c(\omega)} = \frac{1}{c_{l,t}} + \delta_\nu \omega^{\nu-1} \sin\left(\frac{\nu\pi}{2}\right).$$

Multivalued solutions are predicted by original model equation in time domain for $\nu > 1$ and sufficiently small δ_ν. In the left plot $a = \alpha(\omega)x_{\mathrm{sh}}$ and x_{sh} is the shock formation distance:



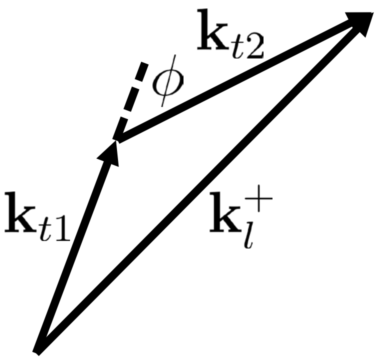


Cormack and Hamilton (Wave Motion 2019)

Noncollinear wave interaction in an isotropic solid

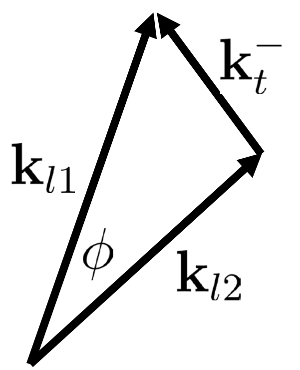
$$\mathbf{k}_{t1} + \mathbf{k}_{t2} = \mathbf{k}_l^+$$

$$\omega_1 + \omega_2 = \omega_+$$



$$\mathbf{k}_{l1} - \mathbf{k}_{l2} = \mathbf{k}_t^-$$

$$\omega_1 - \omega_2 = \omega_-$$



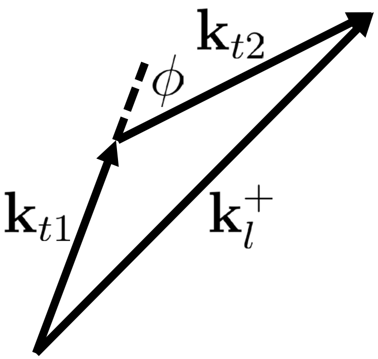
Interaction cases which produce a scattered wave.[a]

Primary waves	Resonant wave type and frequency	Direction of scattered wave	$\cos\phi$[b]	Frequency limits[c]
Two transverse	Longitudinal $(\omega_1+\omega_2)$	$\mathbf{k}_1+\mathbf{k}_2$	$c^2+\frac{(c^2-1)(a^2+1)}{2a}$	$\frac{1-c}{1+c}<a<\frac{1+c}{1-c}$
Two longitudinal	Transverse $(\omega_1-\omega_2)$	$\frac{\mathbf{k}_1-\mathbf{k}_2}{\omega_1-\omega_2}$	$\frac{c^2}{1}+\frac{(c^2-1)(a^2+1)}{2ac^2}$	$\frac{1-c}{1+c}<a<\frac{1+c}{1-c}$
One longitudinal and one transverse[d]	Longitudinal $(\omega_1+\omega_2)$	$\mathbf{k}_1+\mathbf{k}_2$	$c+\frac{a(c^2-1)}{2c}$	$0<a<\frac{2c}{(1-c)}$
One longitudinal and one transverse[d]	Longitudinal $(\omega_1-\omega_2)$	$\frac{\mathbf{k}_1-\mathbf{k}_2}{\omega_1-\omega_2}$	$c+\frac{a(1-c^2)}{2c}$	$0<a<\frac{2c}{(1+c)}$
One longitudinal and one transverse[d]	Transverse $(\omega_1-\omega_2)$	$\frac{\mathbf{k}_1-\mathbf{k}_2}{\omega_1-\omega_2}$	$\frac{1}{c}+\frac{(c^2-1)}{2ac}$	$\frac{1-c}{2}<a<\frac{1+c}{2}$

[a] From Jones and Kobett [9]. (1963)
[b] ϕ is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$ at resonance; a is the frequency ratio ω_2/ω_2; c is velocity ratio c_{tr}/c_{long}.
[c] When a is within the limits shown, it is possible to choose an angle ϕ that will give a scattered wave.
[d] The frequency of the longitudinal primary wave is ω_1.

Noncollinear wave interaction in an isotropic solid

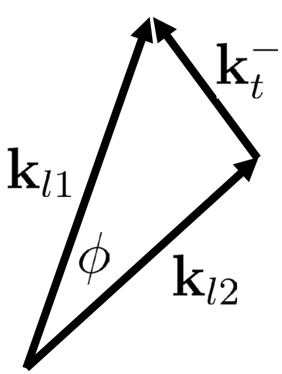
$$\mathbf{k}_{t1} + \mathbf{k}_{t2} = \mathbf{k}_l^+$$
$$\omega_1 + \omega_2 = \omega_+$$



These relations are often expressed as conservation of quasi-momentum and energy of phonons:

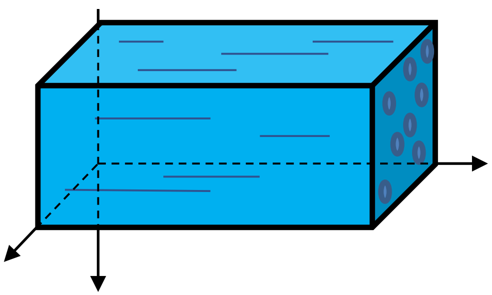
$$\hbar\mathbf{k}_1 + \hbar\mathbf{k}_2 = \hbar\mathbf{k}_3 \qquad \text{quasi-momentum}$$
$$\hbar\omega_1 + \hbar\omega_2 = \hbar\omega_3 \qquad \text{energy}$$

$$\mathbf{k}_{l1} - \mathbf{k}_{l2} = \mathbf{k}_t^-$$
$$\omega_1 - \omega_2 = \omega_-$$

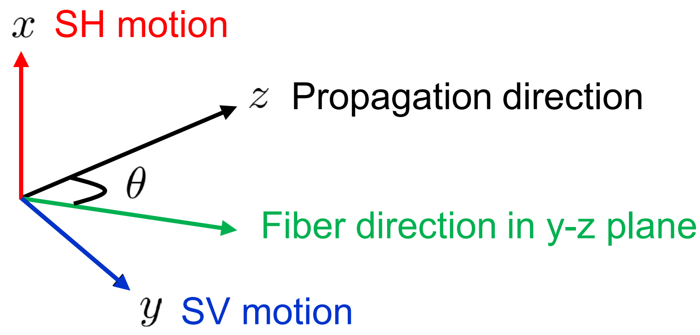


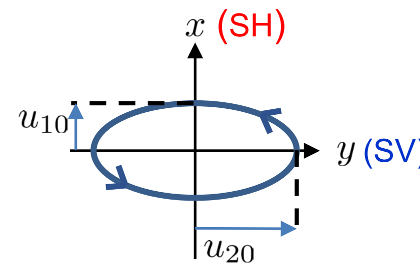
- Noncollinear interaction required for resonant mode conversion because of difference in shear and compressional wave speeds
- Noncollinear interaction required for resonant interaction in optics because of strong dispersion associated with material nonlinearity

Collinear wave interaction in an anisotropic solid

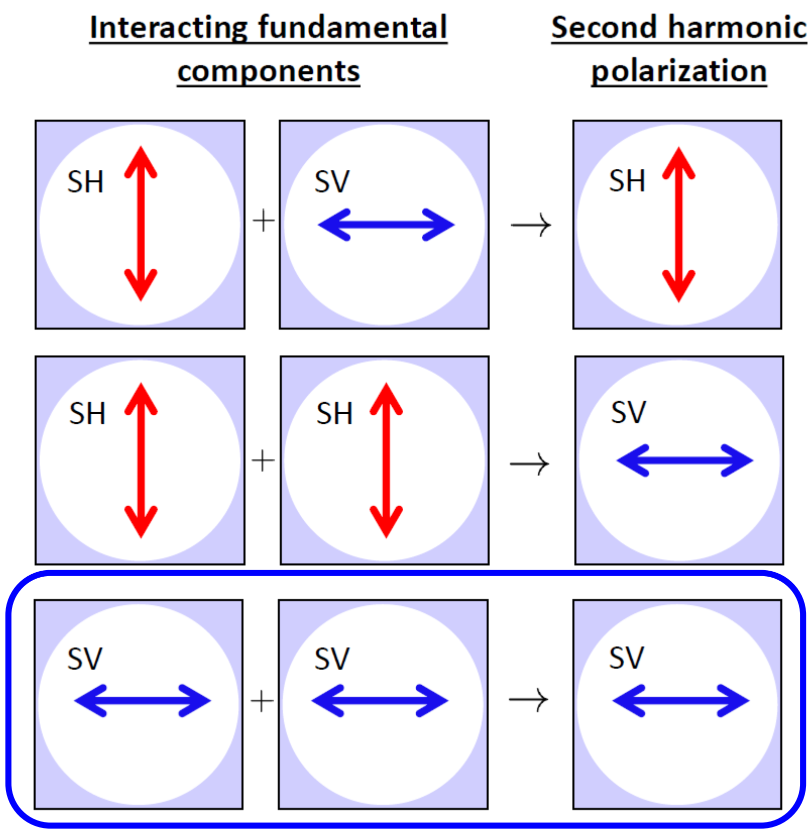


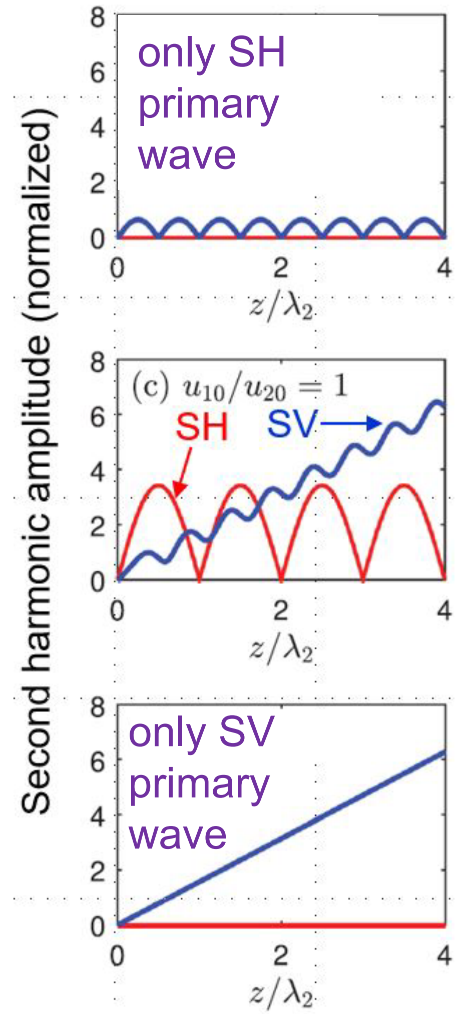
transversely isotropic solid





- Inter-modal interaction is **inefficient** because of difference in propagation speeds
- Second harmonic generation from a source with both SH and SV components:
- Only **SV** + **SV** = **SV** interaction is efficient





Cormack (JASA 2021)

Measurement of TOE constants (*A*,*B*,*C*): Acoustoelasticity

For compressional waves in an isotropic solid there are 5 elastic constants (μ,K,A,B,C) through 3rd order in the strain energy density:

$$\mathcal{E} = \mu I_2 + (\tfrac{1}{2}K - \tfrac{1}{3}\mu)I_1^2 + \tfrac{1}{3}AI_3 + BI_1I_2 + \tfrac{1}{3}CI_1^3$$

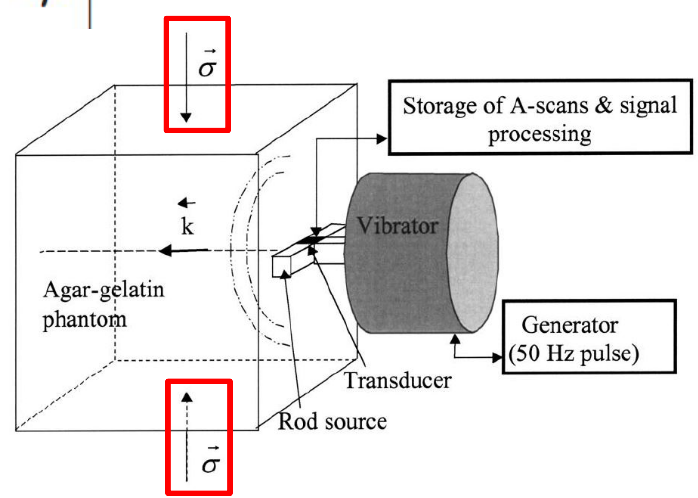
Application of stress σ changes the 3 wave speeds, as a function of TOE constants, relative to the equilibrium state (expressed here in terms of the 2nd order Lamé constants λ and μ instead of μ and K):

$$\rho(V_p)^2 = \lambda + 2\mu - \frac{\sigma}{3\lambda+2\mu}\left[-\frac{\lambda}{\mu}A + 2B\left(1-\frac{\lambda}{\mu}\right) + 2C - 4\lambda - 2\frac{\lambda^2}{\mu}\right]$$

$$\rho(V_s^{\parallel})^2 = \mu - \frac{\sigma}{3\lambda+2\mu}\left[\frac{A}{2}\left(1+\frac{\lambda}{2\mu}\right) + B + \lambda + 2\mu\right]$$

$$\rho(V_s^{\perp})^2 = \mu - \frac{\sigma}{3\lambda+2\mu}\left(-\frac{A\lambda}{2\mu} + B - 2\lambda\right)$$

The 3 wave speeds (1 compressional and 2 shear) permit determination of (*A*,*B*,*C*)



Catheline, Gennisson, and Fink (JASA 2003)

Measurement of TOE constants (*A*,*B*,*C*): Results for soft elastic media

TABLE I. Elastic moduli measured in three Agar-gelatin-based phantoms.

$(K = \lambda + \frac{2}{3}\mu)$	Linear second-order elastic moduli (Lamé coefficients)		Nonlinear third order elastic moduli (Landau coefficients)		
Phantom #	λ (GPa)	μ (kPa)	A (kPa)	B (GPa)	C (GPa)
1	2.25	9.0 ± 0.2	-64 ± 13	-12 ± 3	24 ± 6
2	2.25	6.35 ± 0.04	-101 ± 7	-14 ± 2	31 ± 3
3	2.25	9.67 ± 0.06	-68 ± 3	-26 ± 2	67 ± 4

Catheline, Gennisson, and Fink (JASA 2003)

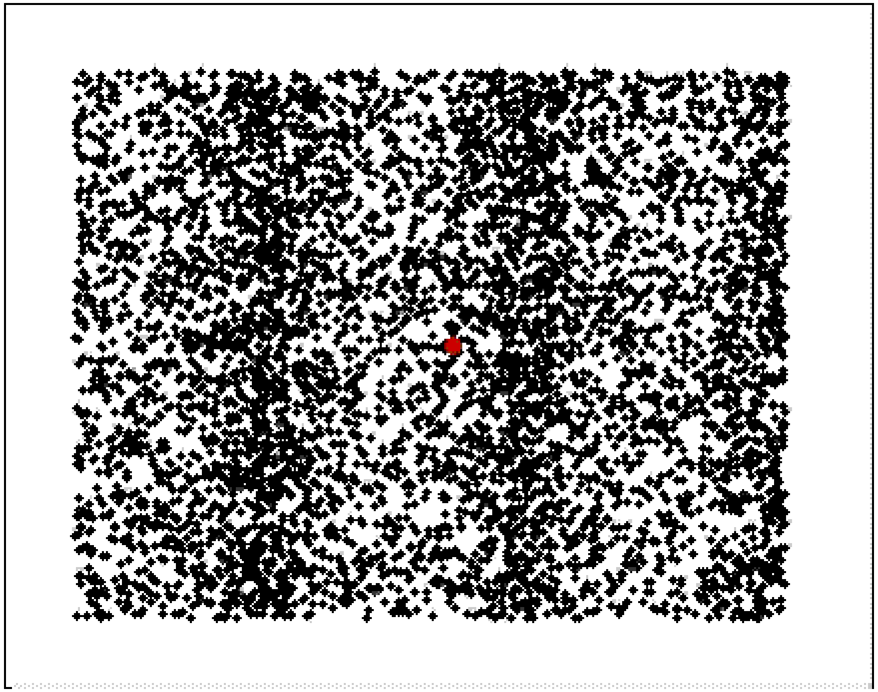
From these results one observes that for soft elastic media

$$A = O(\mu) \qquad O\left(\frac{\mu, A}{K, B, C}\right) \sim 10^{-5}$$

These relations are used to simplify the coefficient of nonlinearity for shear waves

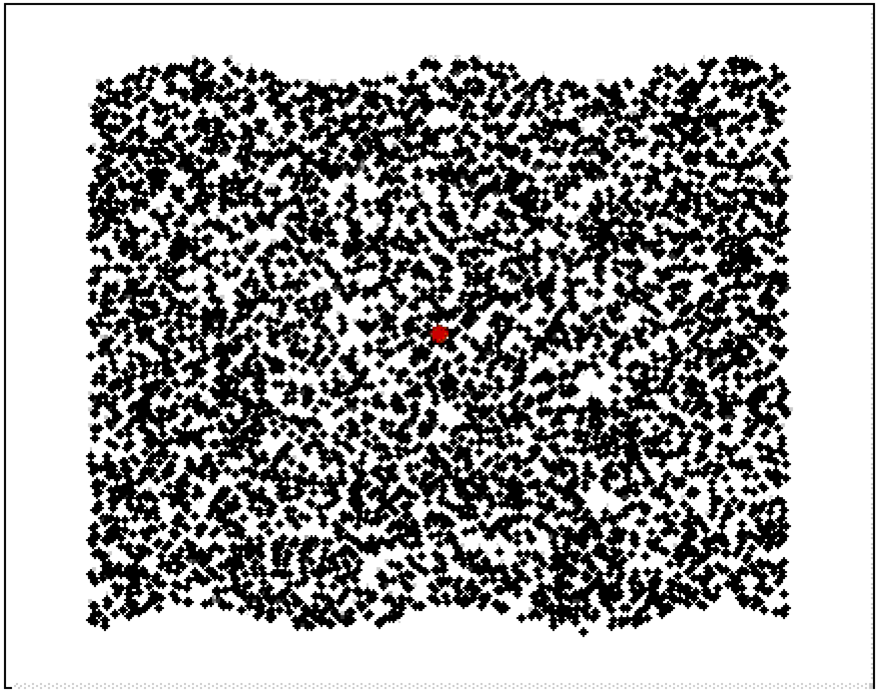
Compressional versus shear waves

Compressional Wave



Compression and rarefaction (“positive and negative”) phases have different physical properties that are affected differently by quadratic nonlinearity

Shear Wave



“Positive and negative” phases have the same physical properties apart from “up versus down,” a symmetry that causes quadratic nonlinearity to have no effect on plane waves

https://acoustics.byu.edu/animations-propagation#poissons-ratio

Shear waves: Elastic energy density required at fourth order

For shear waves to account for cubic nonlinearity, the strain energy density requires 9 independent elastic constants through fourth order (Zabolotskaya 1986):

$$\begin{aligned}\mathcal{E} = \mu I_2 &+ (\tfrac{1}{2}K - \tfrac{1}{3}\mu)I_1^2 && \text{2nd order in strain} \\ &+ \tfrac{1}{3}AI_3 + BI_1I_2 + \tfrac{1}{3}CI_1^3 && \text{3rd order in strain} \\ &+ EI_1I_3 + FI_1^2I_2 + GI_2^2 + HI_1^4 && \text{4th order in strain}\end{aligned}$$

For a plane shear wave the cubic Burgers equation is (see Lee-Bapty and Crighton 1987)

$$\frac{\partial v_x}{\partial z} = \frac{\beta_t}{c_t^3} \boxed{v_x^2 \frac{\partial v_x}{\partial \tau}} + \frac{\delta}{2c_t^3}\frac{\partial^2 v_x}{\partial \tau^2} \qquad \tau = t - z/c_t$$

and the coefficient of nonlinearity is

$$\beta_t = \frac{3}{4\mu}\left[K + \frac{4}{3}\mu + A + 2B + 2G - \frac{(K + \frac{4}{3}\mu + \frac{1}{2}A + B)^2}{K + \frac{1}{3}\mu}\right]$$

Shear waves in soft elastic media

Again, the energy density through fourth order is

$$\mathcal{E} = \mu I_2 + (\tfrac{1}{2}K - \tfrac{1}{3}\mu)I_1^2$$ 2nd order in strain

$$+ \tfrac{1}{3}AI_3 + BI_1I_2 + \tfrac{1}{3}CI_1^3$$ 3rd order in strain

$$+ EI_1I_3 + FI_1^2I_2 + GI_2^2 + HI_1^4$$ 4th order in strain

For soft elastic media (e.g., tissue) it was found that (Catheline et al. 2003)

$$A = O(\mu) \qquad O\left(\frac{\mu, A}{K, B, C}\right) \sim 10^{-5}$$

For shear waves in soft elastic media only 3 elastic constants are needed, 1 at each order:

$$\mathcal{E} = \mu I_2 + \tfrac{1}{3}AI_3 + DI_2^2$$

The coefficient of nonlinearity for a plane shear wave then reduces to

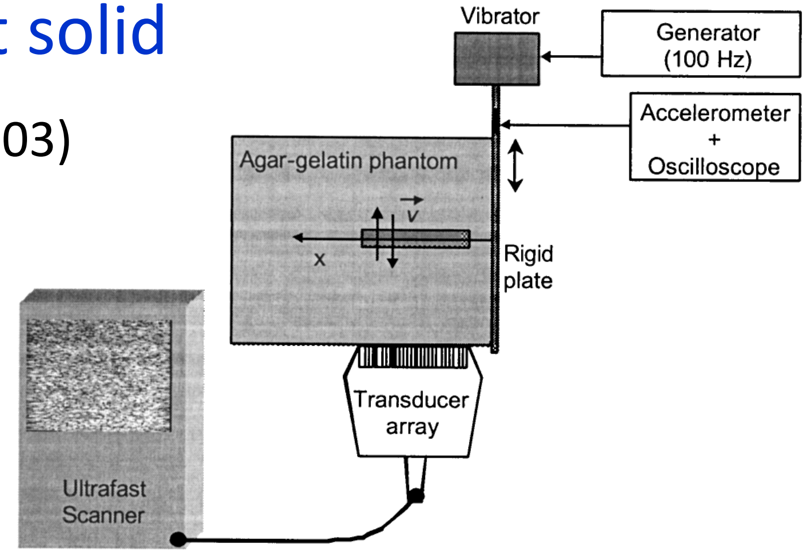
$$\beta_t = \frac{3}{2}\left(1 + \frac{\frac{1}{2}A + D}{\mu}\right)$$

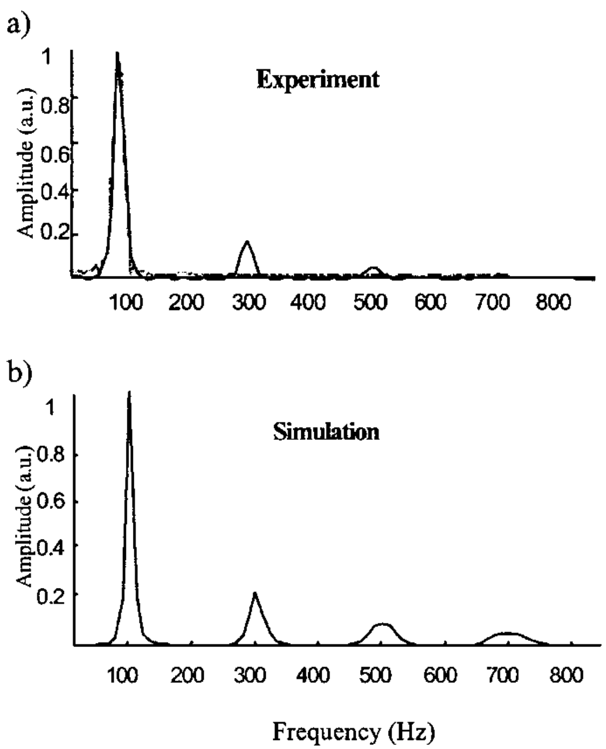
Hamilton, Ilinskii, and Zabolotskaya (JASA 2004)

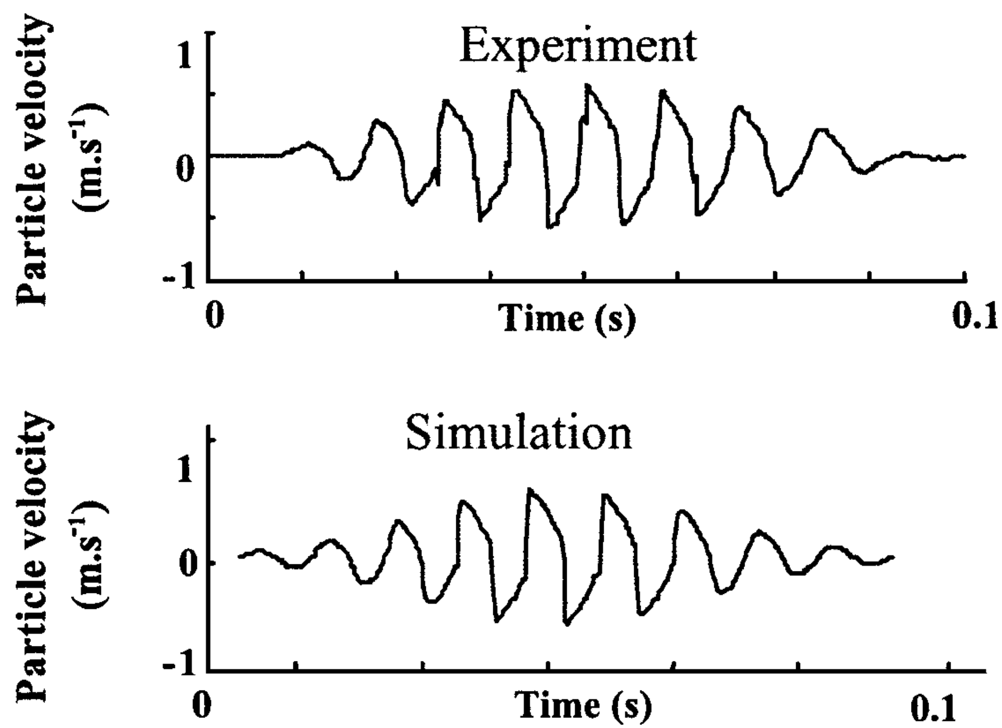
Experimental confirmation in soft solid

Catheline, Gennisson, Tanter, and Fink (PRL 2003)

Source frequency: 100 Hz
Source amplitude: 0.6 m/s
Wave speed: 1.6 m/s
 (acoustic Mach number = 0.38)
Measurement distance: 15 mm
Nonlinearity is ***cubic***







Shear wave beams

Coupled equations for a shear wave beam are

cubic terms

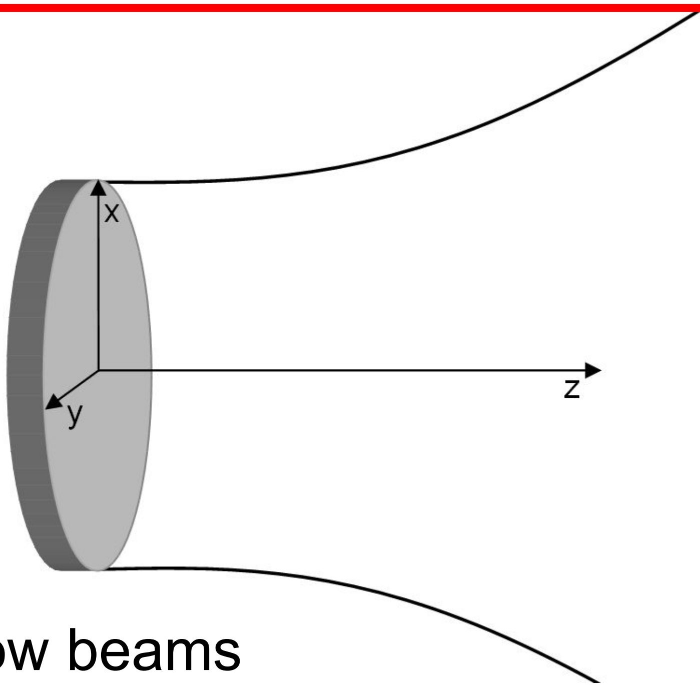
$$\frac{\partial^2 u_x}{\partial z \partial \tau} = \frac{c_t}{2}\nabla_\perp^2 u_x + \frac{1}{2c_t}\left(1+\frac{A}{4\mu}\right)Q_{xy} + \frac{\beta_t}{3c_t^3}\frac{\partial}{\partial \tau}\left\{\frac{\partial u_x}{\partial \tau}\left[\left(\frac{\partial u_x}{\partial \tau}\right)^2 + \left(\frac{\partial u_y}{\partial \tau}\right)^2\right]\right\}$$

$$\frac{\partial^2 u_y}{\partial z \partial \tau} = \frac{c_t}{2}\nabla_\perp^2 u_y + \frac{1}{2c_t}\left(1+\frac{A}{4\mu}\right)Q_{yx} + \frac{\beta_t}{3c_t^3}\frac{\partial}{\partial \tau}\left\{\frac{\partial u_y}{\partial \tau}\left[\left(\frac{\partial u_x}{\partial \tau}\right)^2 + \left(\frac{\partial u_y}{\partial \tau}\right)^2\right]\right\}$$

where *u* is particle displacement.

$$Q_{xy} = \frac{\partial^2 u_y}{\partial \tau^2}\frac{\partial u_y}{\partial x} + \frac{\partial^2 u_y}{\partial \tau^2}\frac{\partial u_x}{\partial y} + 2\frac{\partial^2 u_x}{\partial y \partial \tau}\frac{\partial u_y}{\partial \tau} - \frac{\partial^2 u_y}{\partial y \partial \tau}\frac{\partial u_x}{\partial \tau} - 2\frac{\partial^2 u_x}{\partial \tau^2}\frac{\partial u_y}{\partial y} - \frac{\partial^2 u_y}{\partial x \partial \tau}\frac{\partial u_y}{\partial \tau}$$

quadratic terms



Q_{xy} = 0 for plane waves, and small for narrow beams

Wochner, Hamilton, Ilinskii, and Zabolotskaya (JASA 2003)

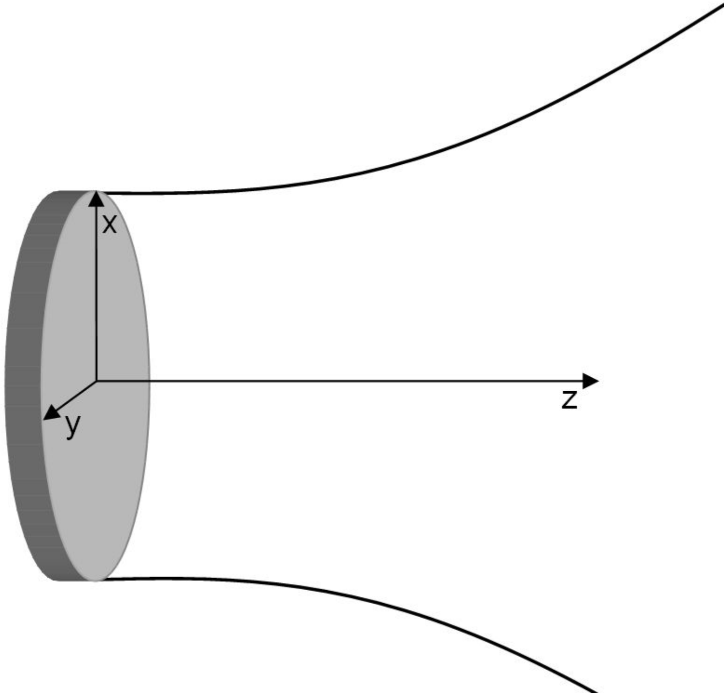
Cubic nonlinearity

Without quadratic nonlinearity (narrow beams), coupled equations assume KZK form:

$$\frac{\partial v_x}{\partial z} = \frac{\beta}{3c^3}\frac{\partial}{\partial \tau}\left[v_x(v_x^2+v_y^2)\right] + \frac{\eta}{2\rho c^3}\frac{\partial^2 v_x}{\partial \tau^2} + \frac{c}{2}\int_{-\infty}^{\tau}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)d\tau'$$

$$\frac{\partial v_y}{\partial z} = \frac{\beta}{3c^3}\frac{\partial}{\partial \tau}\left[v_y(v_x^2+v_y^2)\right] + \frac{\eta}{2\rho c^3}\frac{\partial^2 v_y}{\partial \tau^2} + \frac{c}{2}\int_{-\infty}^{\tau}\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}\right)d\tau'$$

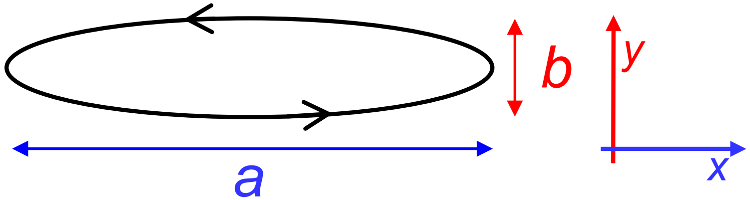
where *v* is particle velocity.

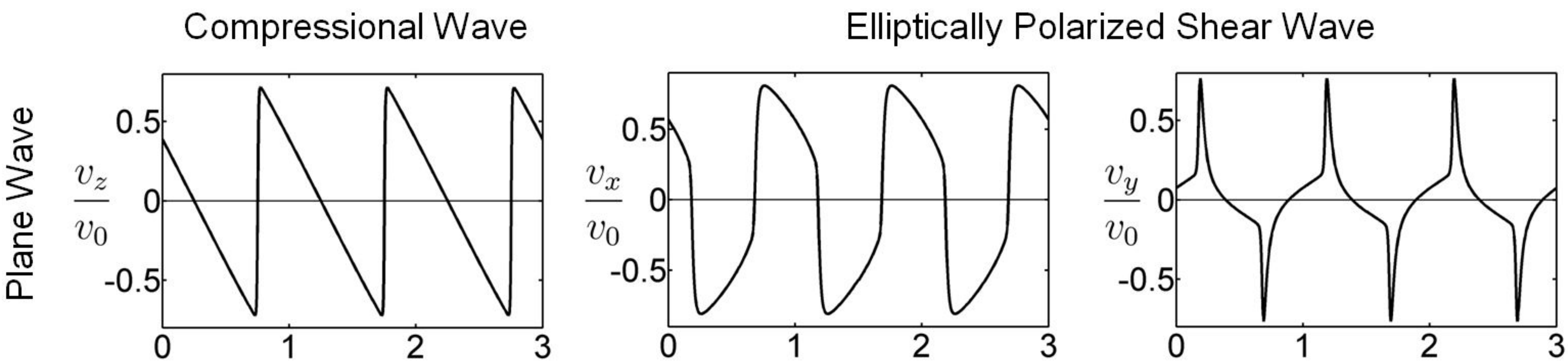


Wochner et al. (JASA 2003)

Elliptical polarization

$b/a = 0.25$

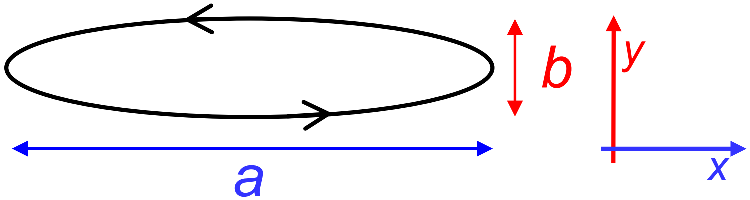


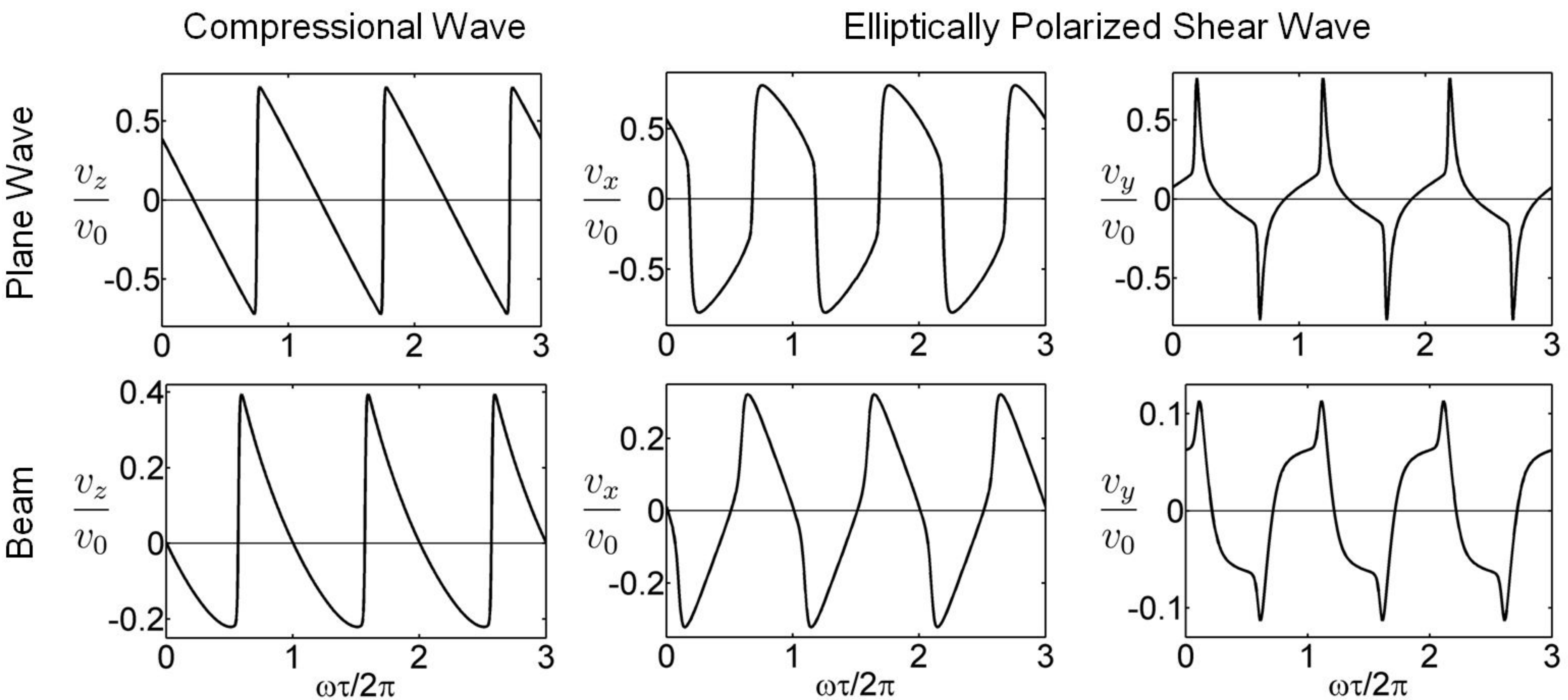


Wochner et al. (JASA 2003)

Elliptical polarization

$b/a = 0.25$

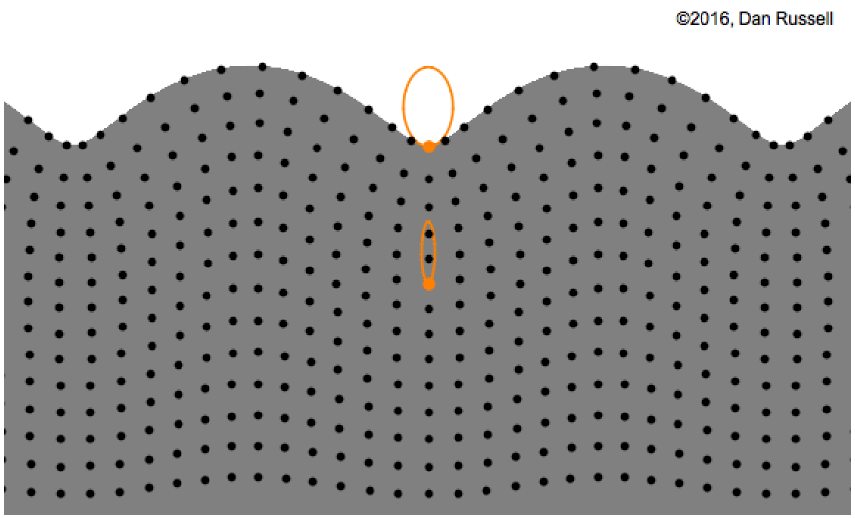




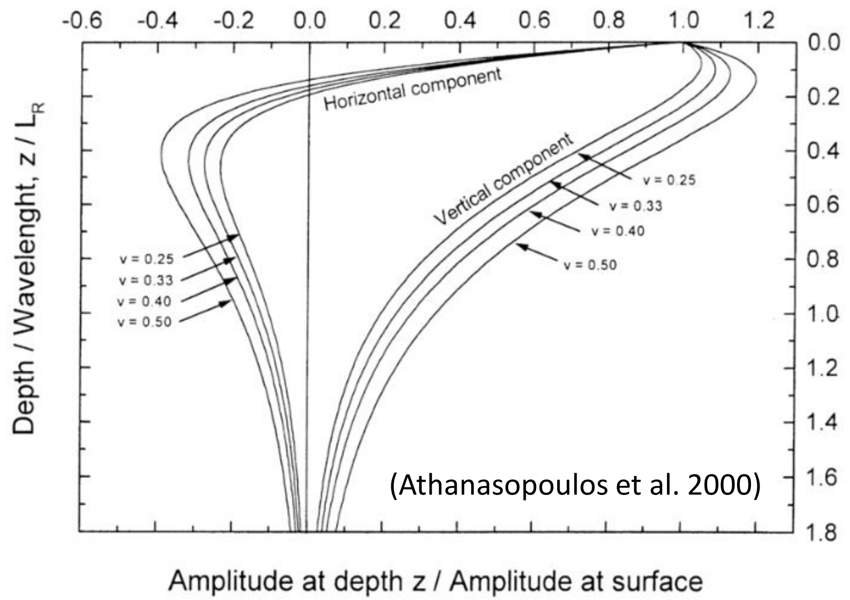
Wochner et al. (JASA 2003)

Surface Acoustic Waves

Nonlinear surface acoustic waves (SAWS) in isotropic solids: Rayleigh waves



https://acoustics.byu.edu/animations-propagation#poissons-ratio



Fourier expansion of the particle displacement fields in the linear approximation:

$$u_x(x,z,t) = \frac{i}{2}\sum_{n=1}^{\infty} u_n(\xi_t e^{n\xi_t kz} + \eta e^{n\xi_l kz})e^{-in\omega_0(t-x/c_R)} + \text{c.c.}$$

$$u_z(x,z,t) = \frac{1}{2}\sum_{n=1}^{\infty} u_n(e^{n\xi_t kz} + \xi_l \eta e^{n\xi_l kz})e^{-in\omega_0(t-x/c_R)} + \text{c.c.}$$

Spectral evolution equation

To account for nonlinearity replace Fourier coefficients in linear solution by slowly varying functions of time $u_n(t)$:

$$u_x(x,z,t) = \frac{i}{2}\sum_{n=1}^{\infty} \underline{u_n(t)}(\xi_t e^{n\xi_t kz} + \eta e^{n\xi_l kz})e^{-in\omega_0(t-x/c_R)} + \text{c.c.}$$

$$u_z(x,z,t) = \frac{1}{2}\sum_{n=1}^{\infty} \underline{u_n(t)}(e^{n\xi_t kz} + \xi_l \eta e^{n\xi_l kz})e^{-in\omega_0(t-x/c_R)} + \text{c.c.}$$

Hamiltonian mechanics is used with $u_n(t)$ interpreted as a generalized displacement to obtain coupled spectral evolution equations for slowly varying particle velocities $v_n(x)$:

$$\frac{dv_n}{dx} = \frac{\mu\omega_0 n^2}{2\rho c_R^4 \zeta}\left(2\underbrace{\sum_{m=n+1}^{N} R_{m,n-m} v_m v_{m-n}^*}_{\text{difference-frequency generation}} - \underbrace{\sum_{m=1}^{n-1} R_{m,n-m} v_m v_{n-m}}_{\text{sum-frequency generation}}\right) - An^2 v_n$$

Zabolotskaya (JASA 1992), Knight et al. (JASA 1997)

Time domain formulations reveal nonlocal nonlinearity

The evolution equation can be expressed in the time domain when evaluated for the horizontal velocity component at the surface $z = 0$:

$$\frac{\partial v_x}{\partial x} = \underbrace{C\frac{\partial v_x^2}{\partial \tau}} + \underbrace{\frac{\partial}{\partial \tau}\iint_T \frac{\Phi(\tau_1/\tau_2)}{\tau_1^2 + \tau_2^2} v_x(x, \tau - \tau_1) v_x(x, \tau - \tau_2)\, d\tau_1 d\tau_2}$$

nonlinearity as in fluids, which is local to instant τ in the waveform

nonlinearity is nonlocal due to integration over convolution of waveform with itself

It can also be expressed using Hilbert transforms:

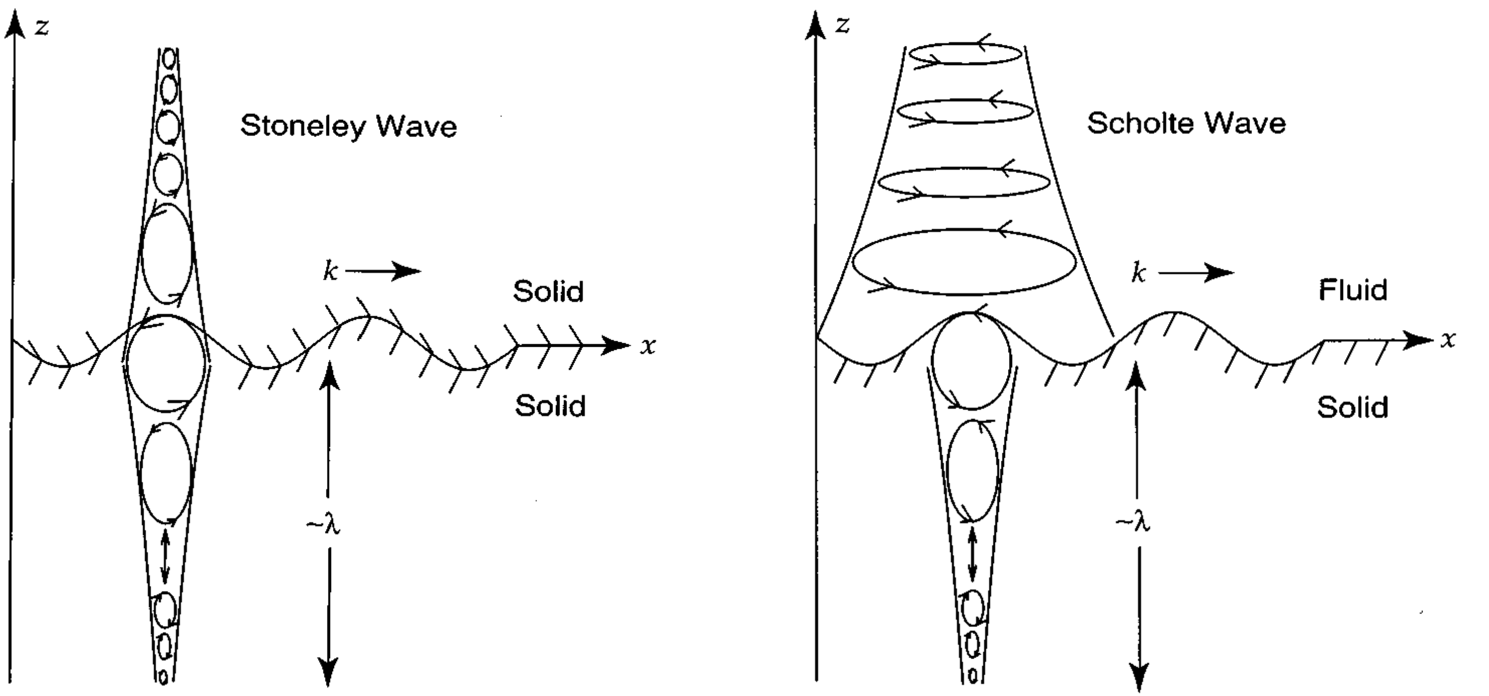
$$\frac{\partial U_x}{\partial X} = \frac{1}{4}\frac{\partial^2 U_x^2}{\partial \theta^2} + \frac{1}{2}\mathcal{H}\left[U_x \mathcal{H}\left[\frac{\partial^2 U_x}{\partial \theta^2}\right]\right], \quad V_x = \frac{\partial U_x}{\partial \theta}$$

where

$$\mathcal{H}[f] = \frac{1}{\pi}\,\mathrm{Pr}\int_{-\infty}^{\infty} \frac{f(t')}{t' - t}\, dt'$$

Hamilton, Ilinskii, and Zabolotskaya (JASA 1995)

Energy method facilitates generalization to Stoneley and Scholte waves



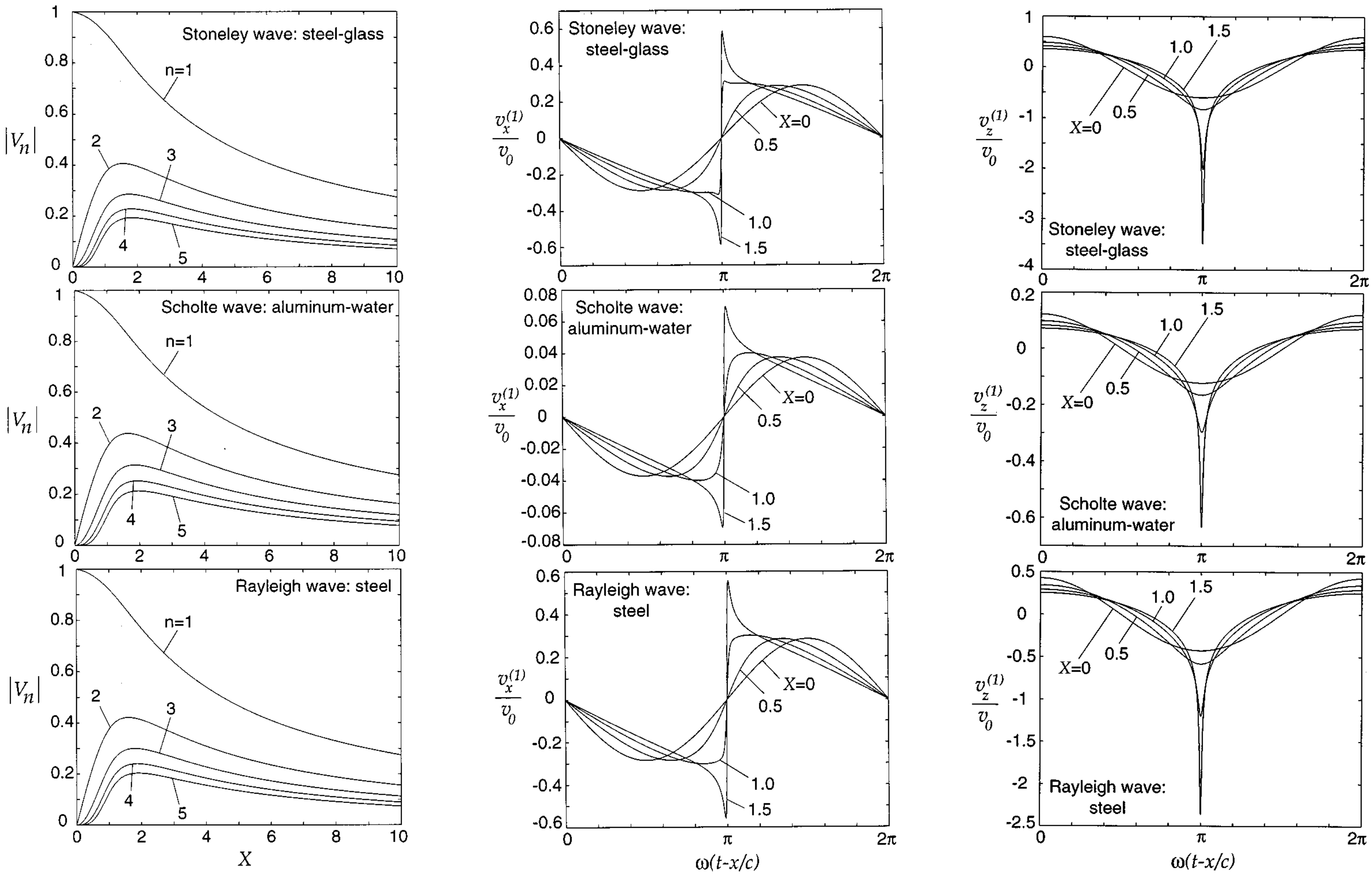
Functional form of spectral evolution equation is unaltered:

$$\frac{dv_n}{dx} = \frac{\mu\omega_0 n^2}{2\rho c_R^4 \zeta}\left(2\sum_{m=n+1}^{N} R_{m,n-m} v_m v_{m-n}^* - \sum_{m=1}^{n-1} R_{m,n-m} v_m v_{n-m}\right) - An^2 v_n$$

Meegan, Hamilton, Ilinskii, and Zabolotskaya (JASA 1999)

Nonlinear surface acoustic waves (SAWs)

harmonic propagation curves | horizontal velocity waveforms | vertical velocity waveforms



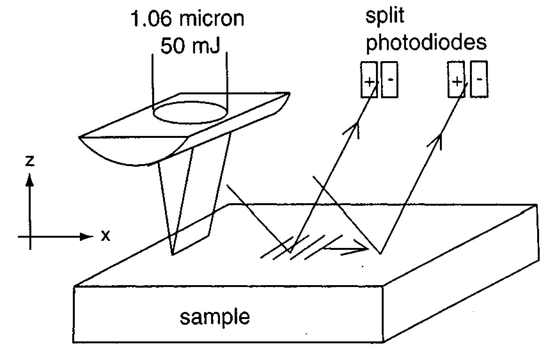
Meegan et al. (JASA 1999)

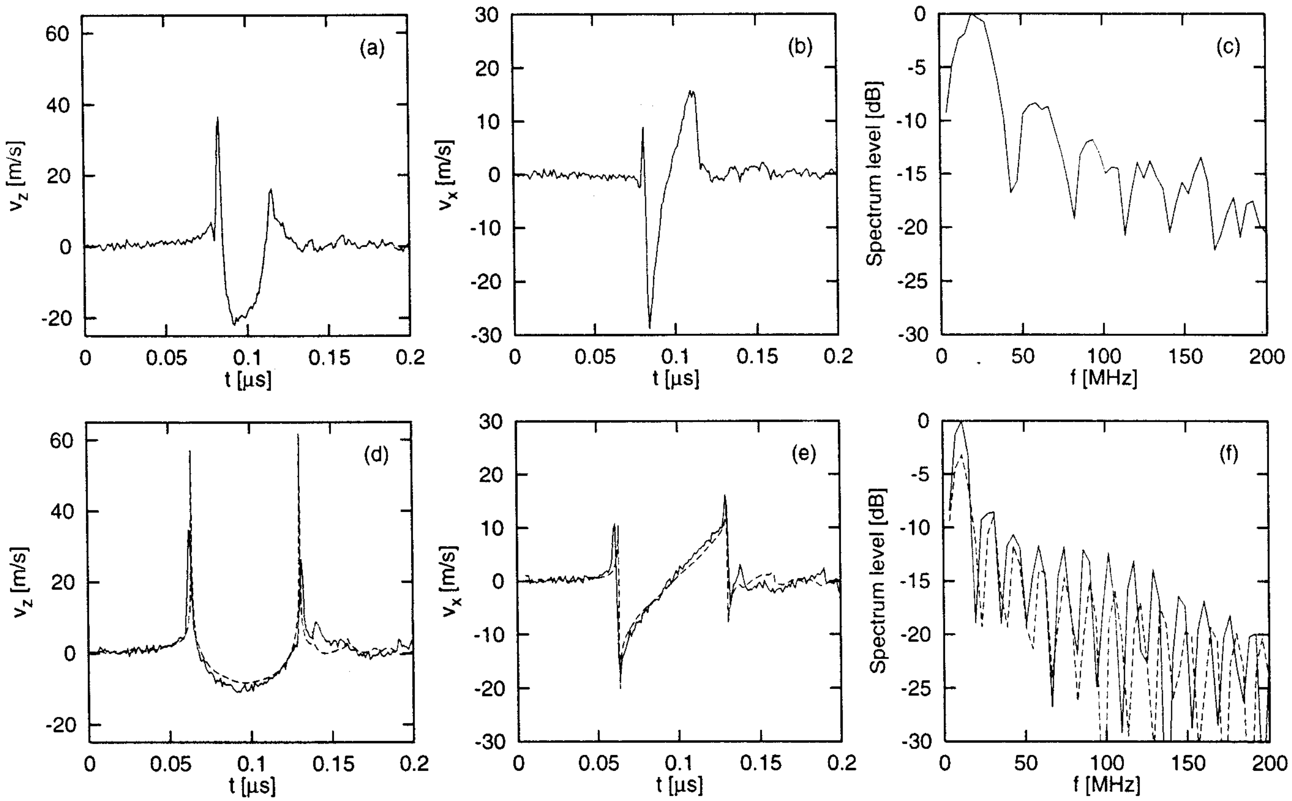
Experimental confirmation in fused quartz

Laser-generated nonlinear Rayleigh waves with shocks

A. Lomonosov and V. G. Mikhalevich
General Physics Institute, Russian Academy of Sciences, 117942 Moscow, Russia
P. Hess
Institute of Physical Chemistry, University of Heidelberg, D-69120 Heidelberg, Germany
E. Yu. Knight
Department of Physics, University of California—Berkeley, Berkeley, California 94720-7300
M. F. Hamilton and E. A. Zabolotskaya
Department of Mechanical Engineering, The University of Texas at Austin, Austin, Texas 78712-1063

2093 J. Acoust. Soc. Am. **105** (4), April 1999



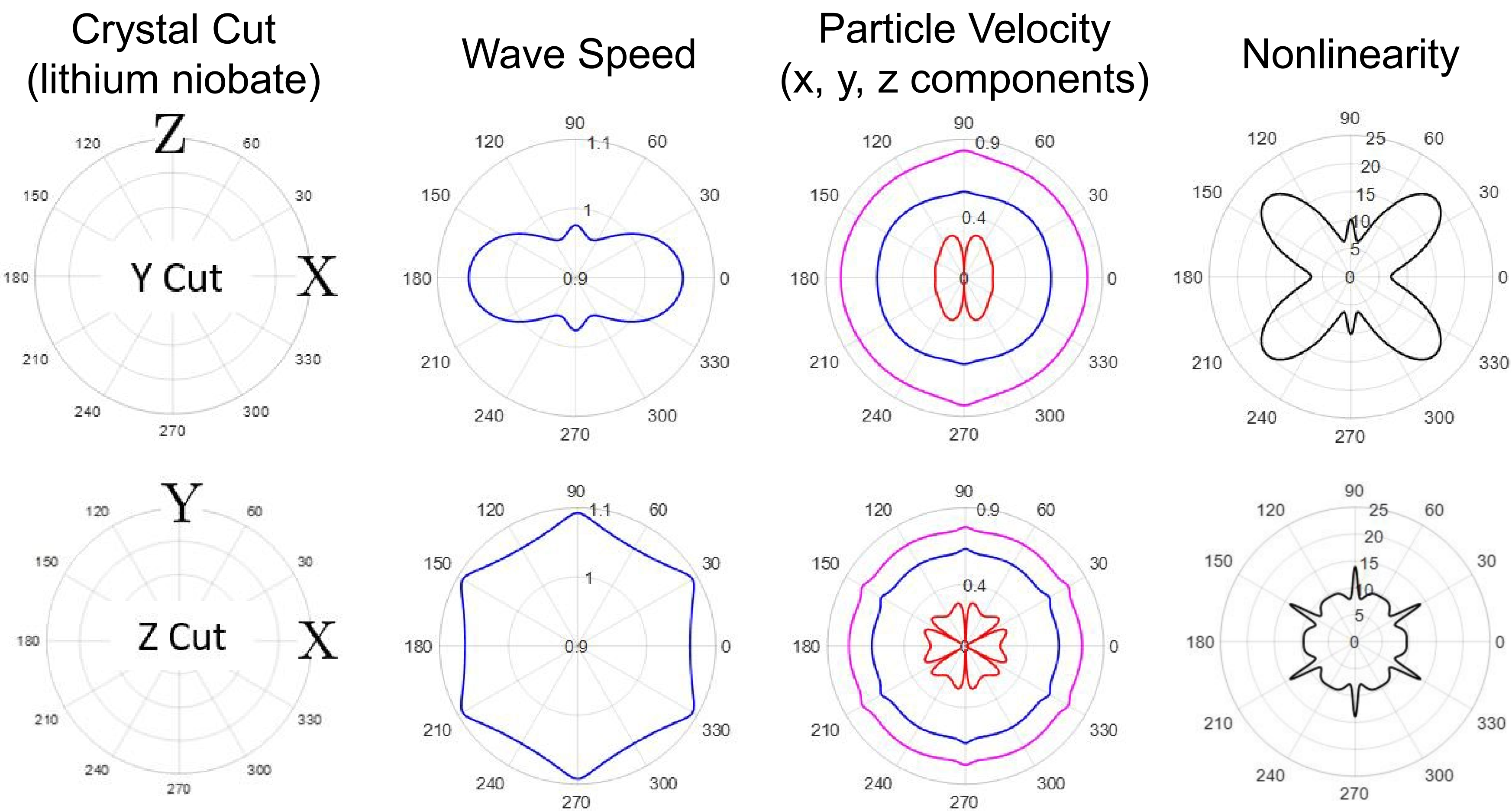


$$f \sim 20 \text{ MHz}$$

$$\overline{x} \simeq 3 \text{ mm}$$

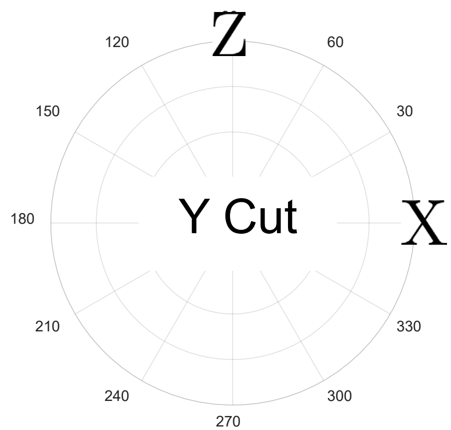
Nonlinear piezoelectric surface acoustic waves (crystals)

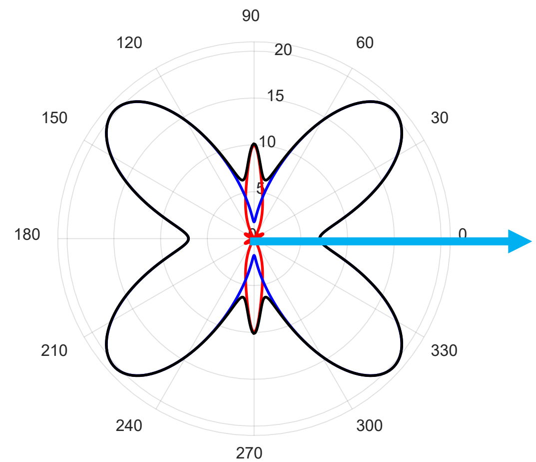
Cormack, Ilinskii, Zabolotskaya, and Hamilton (JASA 2022)

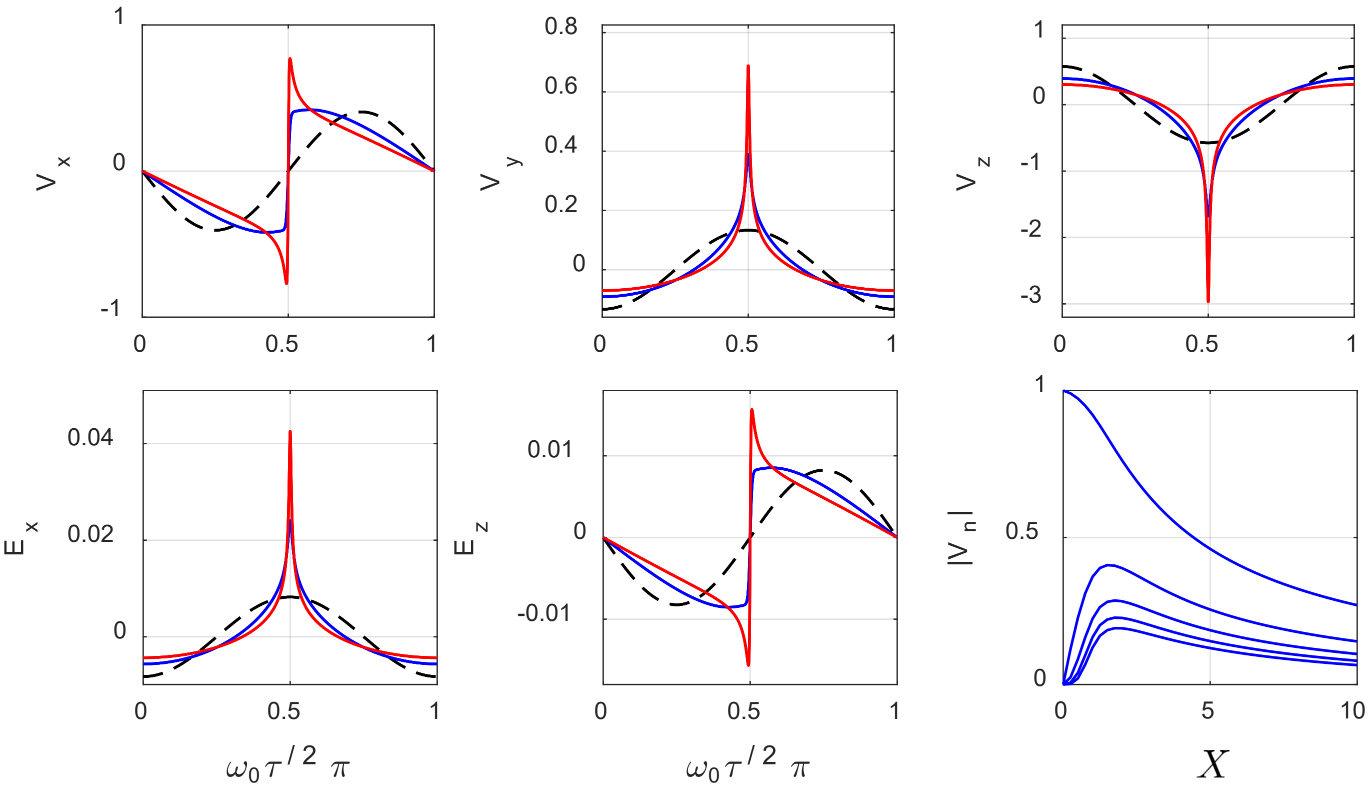


Anisotropy dramatically affects both linear and nonlinear properties

Propagation along X axis (Y cut)







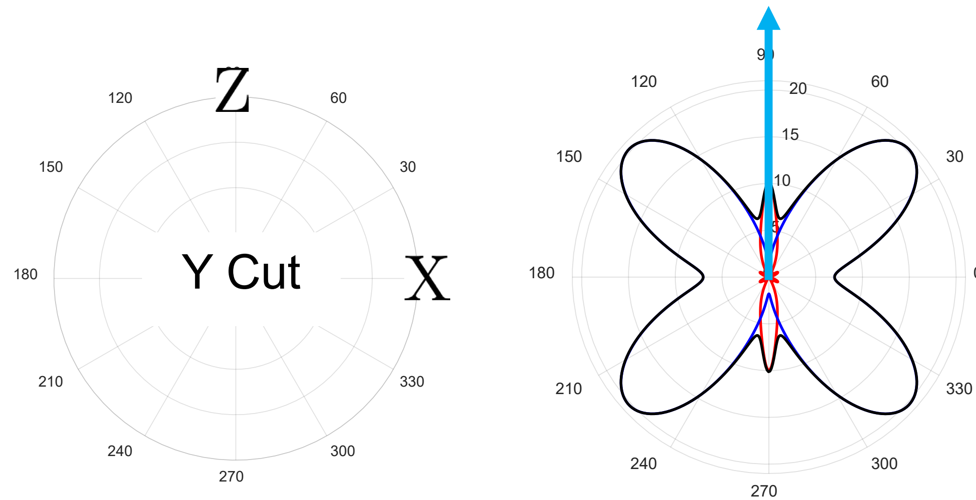
Waveforms plotted at:

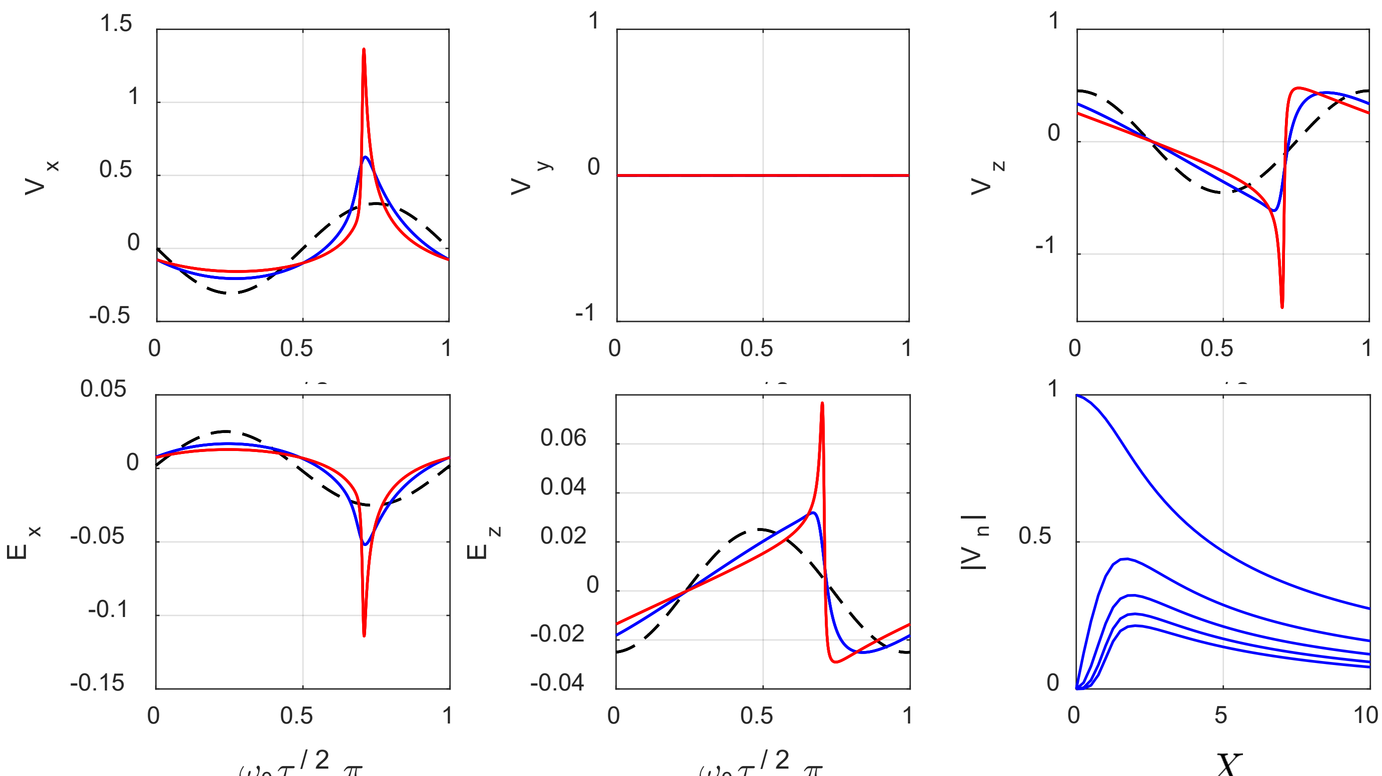
$X = 0$, 1, and 2

where

$$X = \frac{4|S_{11}|\omega_0 v_0}{\rho_0 c_R^4} x$$

Propagation along Z axis (Y cut)





Waveforms plotted at:

$X = 0$, 1, and 2

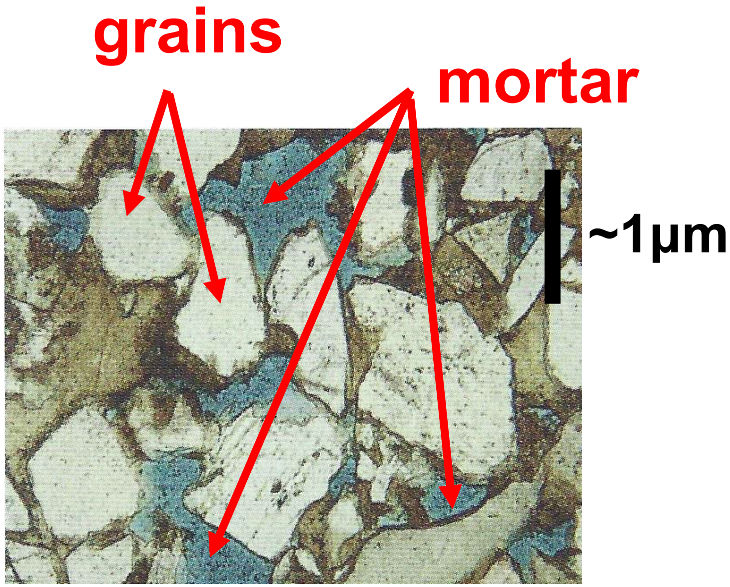
where

$$X = \frac{4|S_{11}|\omega_0 v_0}{\rho_0 c_R^4} x$$

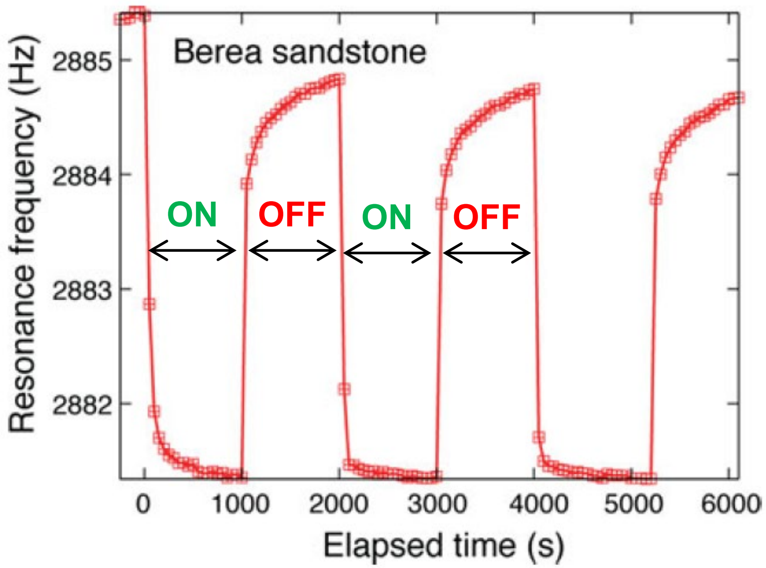
Nonclassical Elastic Media: Sandstone

Nonlinear elasticity of sandstone

- Sandstone microstructure composed of soft mortar binding stiff grains



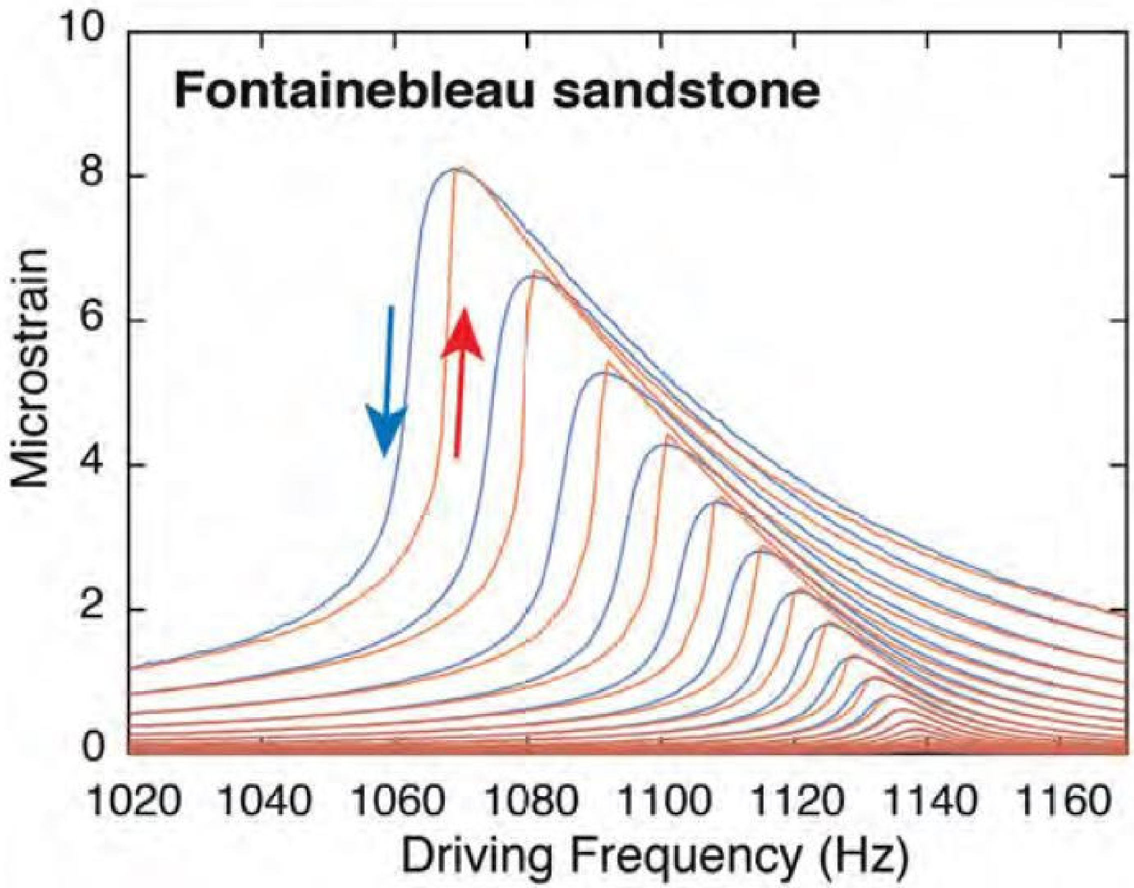
- Previous studies probed sandstone bars using resonance techniques and strains of order 1 μstrain
- Observed elastic softening (~1%) and "slow dynamics"



TenCate (Pure Appl. Geophys. 2011)

Slow dynamics

Traditionally studied in the frequency domain with resonance curves

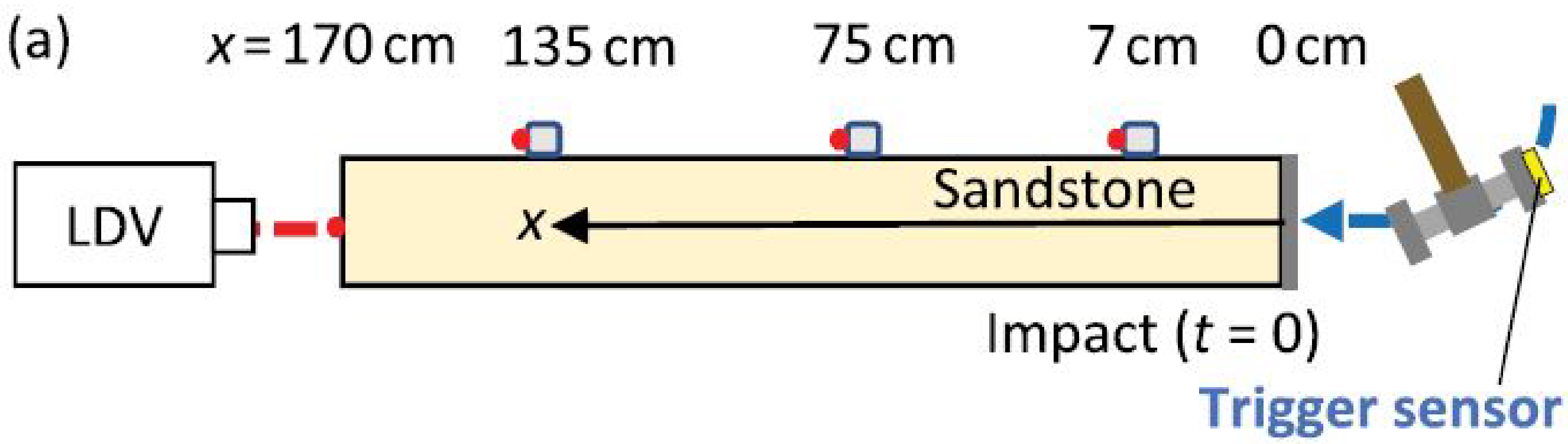


- Resonance curves are history dependent
- Memory manifests as softening

TenCate and Remillieux (Acoustics Today 2019)

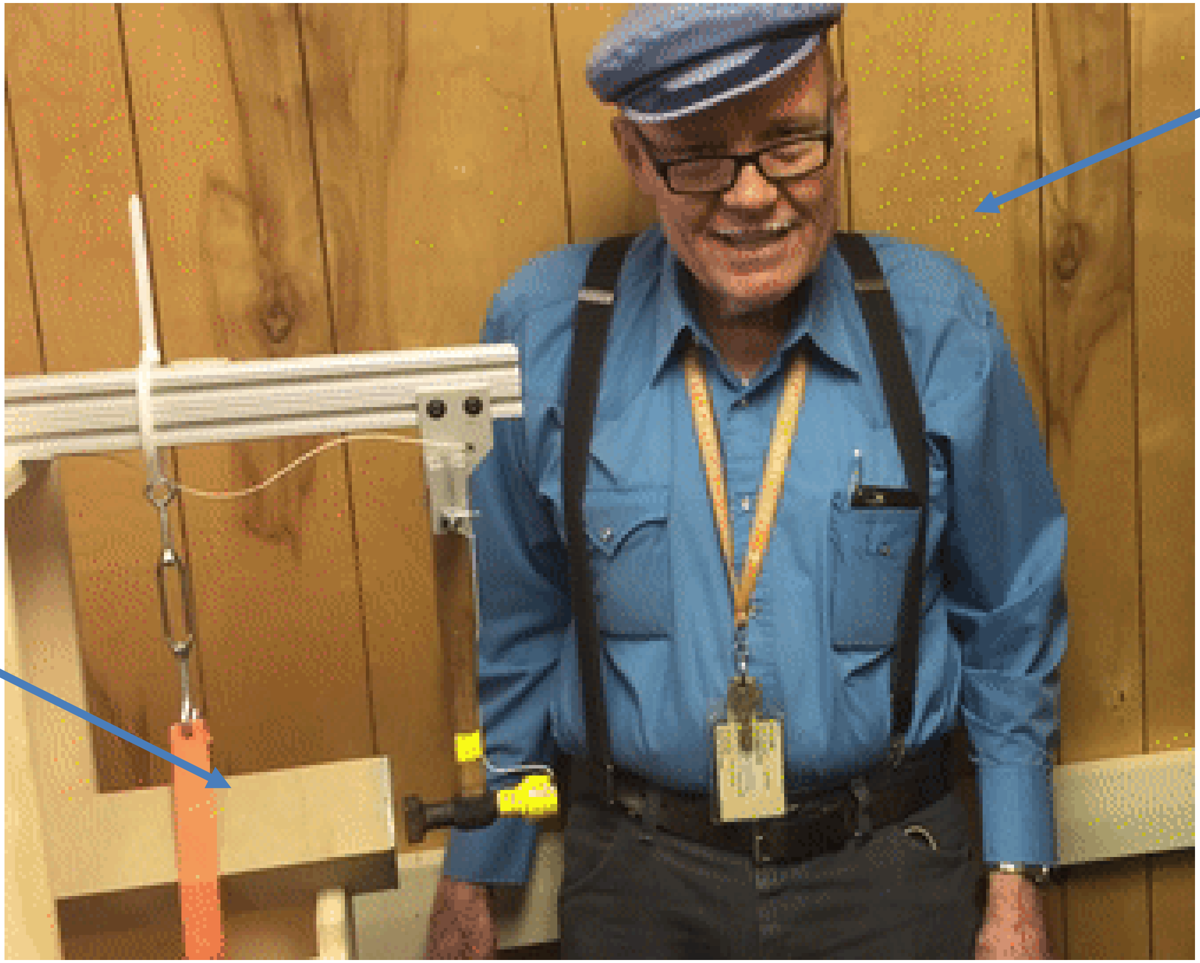
Slow dynamics (time domain)

- Pendulum hammer generates large-amplitude longitudinal waves
- Laser Doppler Vibrometer measures longitudinal particle velocity from **flat end of rod** and **side-mounted reflectors**



Muir, Cormack, Slack, and Hamilton (JASA 2020)

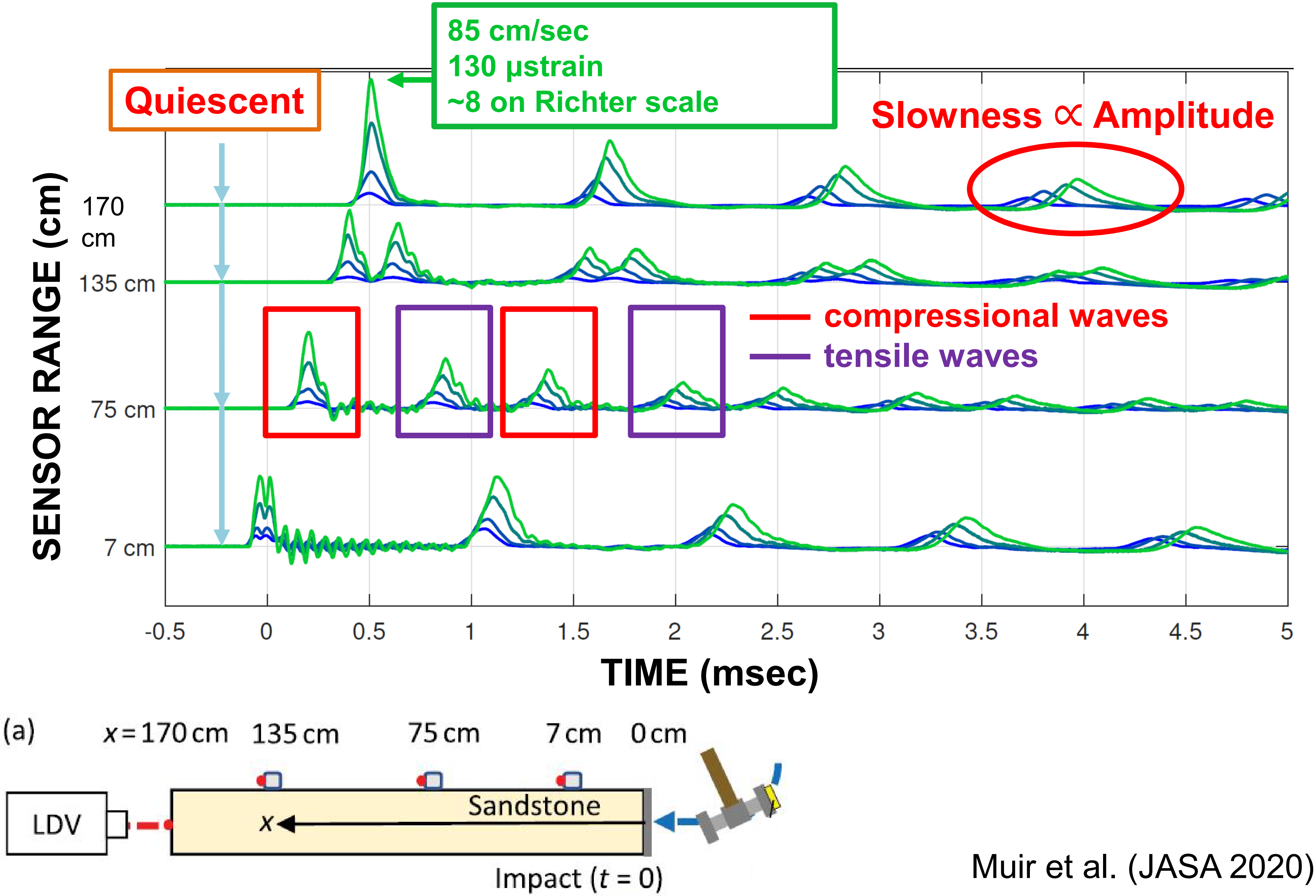
Pulse amplitude controlled by hammer drop height



Thomas Muir

End of sandstone bar

Hammer dropped from progressively greater heights

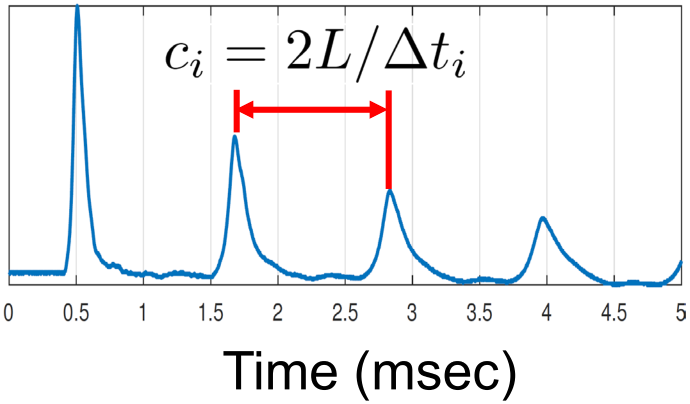


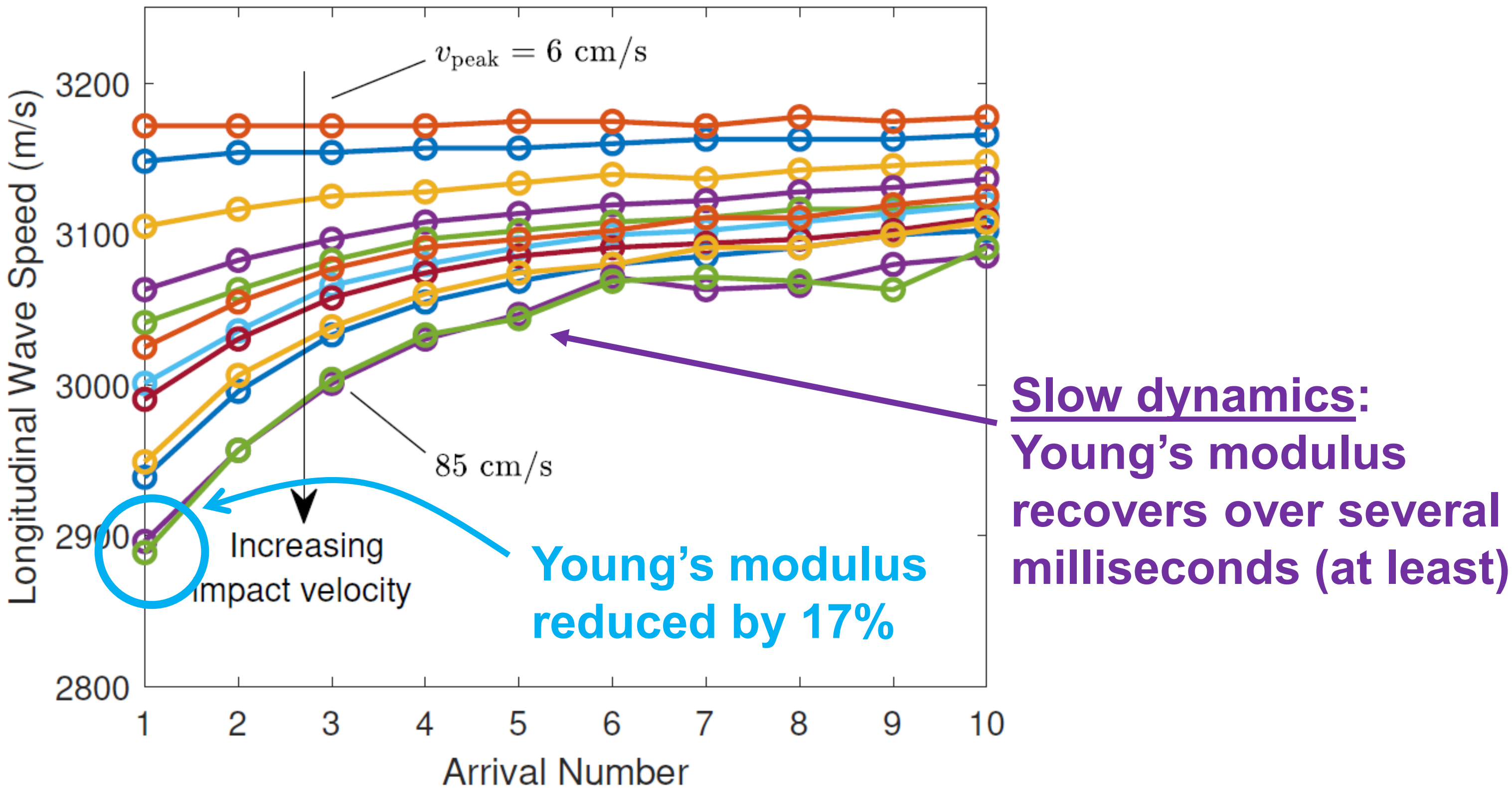
Muir et al. (JASA 2020)

Elastic softening inferred from time-of-flight calculation

- Longitudinal wave speed computed from time between arrivals at far end
- Wave speed related to Young's modulus E:

$$c_B = \sqrt{E/\rho}$$

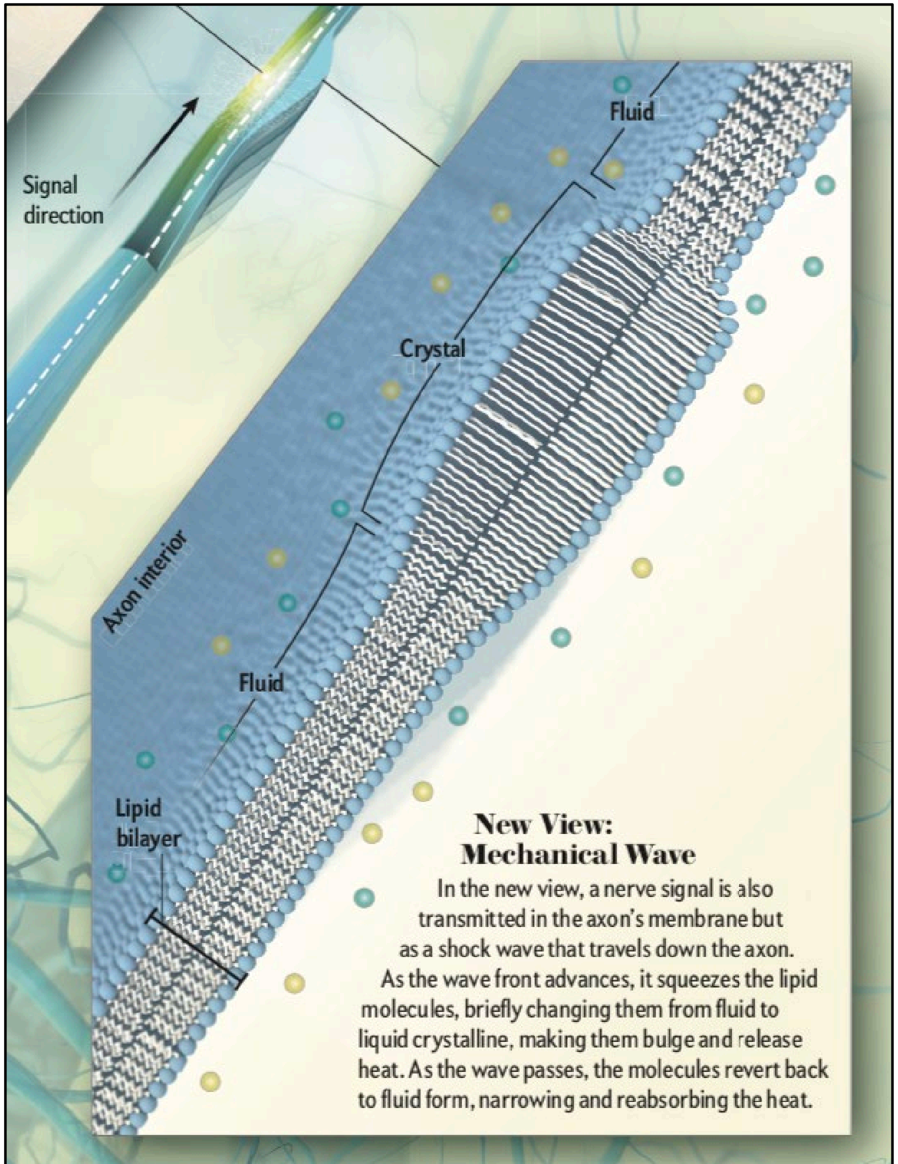




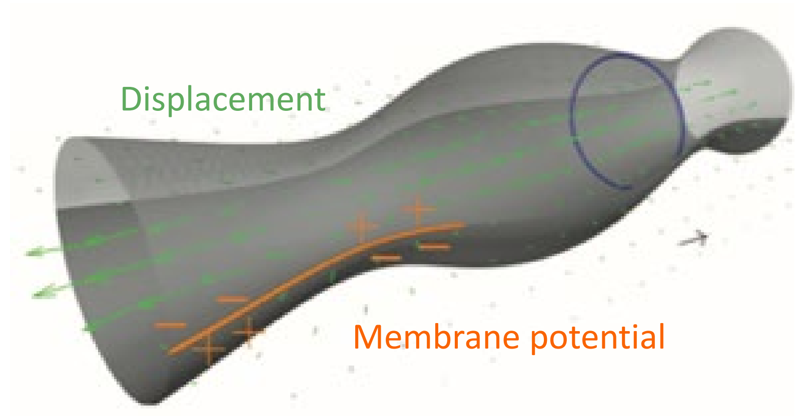
Muir et al. (JASA 2020)

Fractional (Lucassen) Surface Waves

Mechanical waves in the brain



Fox (Sci. Amer. 2018)



El Hady and Machta
(Nat. Commun. 2015)

"The existence of these [mechanical] effects is not in doubt," says Simon Laughlin, a neuroscientist at the University of Cambridge. "The question is whether neurons actually use them to do something useful." (2018)

Linear Lucassen wave (Trans. Faraday Soc. 1968)

- Assume incompressible, viscous, liquid half-space bounded by a thin elastic layer
- Linear model for propagation of the surface wave:

$$K_{2D}\frac{\partial^2 u}{\partial x^2} = \sqrt{\rho\mu}\frac{\partial^{3/2} u}{\partial t^{3/2}}$$

$$u = \text{particle displacement}$$

$$\rho = \text{density of liquid}$$

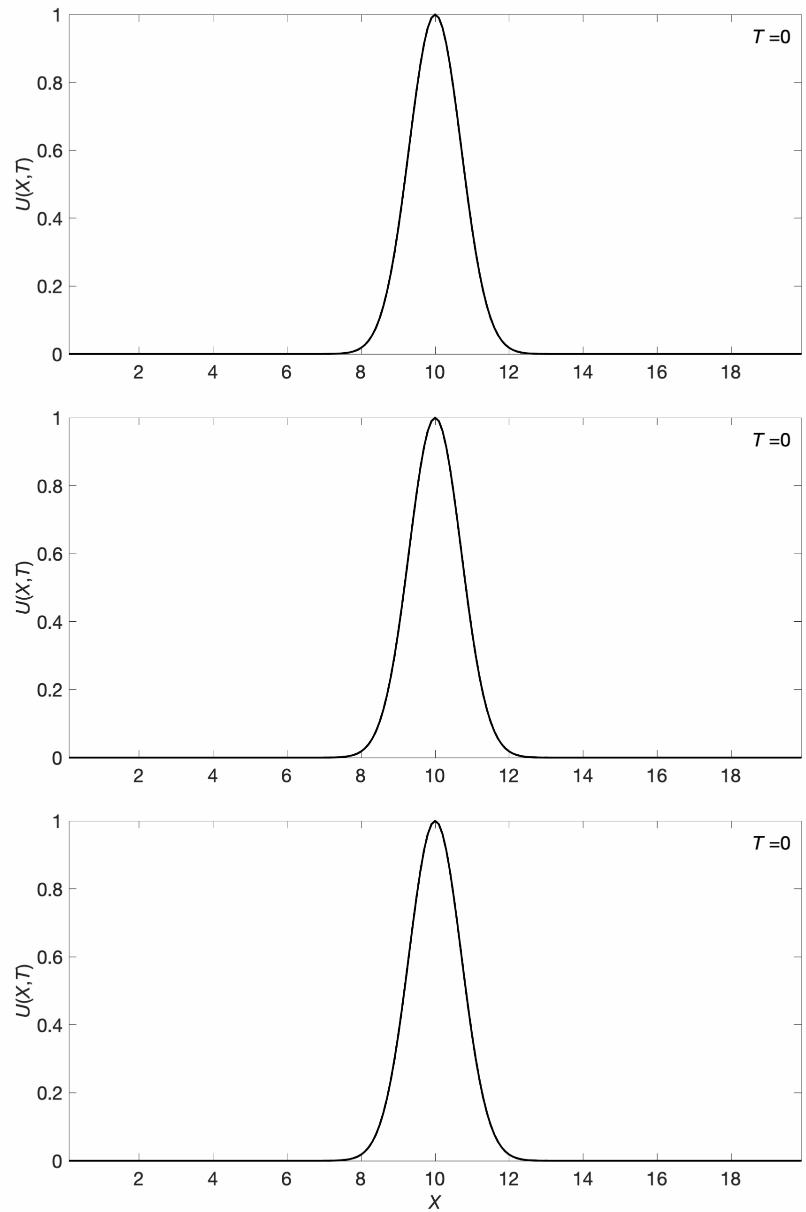
$$\mu = \text{viscosity of liquid}$$

$$K_{2D} = \text{elastic modulus of layer}$$

$$\rho_{2D} = \text{mass density of layer}$$

Kappler, Shrivastava, Schneider, and Netz (Phys. Rev. Fluids 2017)

Visualizing fractional operators



Wave propagation

$$\frac{\partial^2 U}{\partial X^2} = \frac{\partial^2 U}{\partial T^2}$$

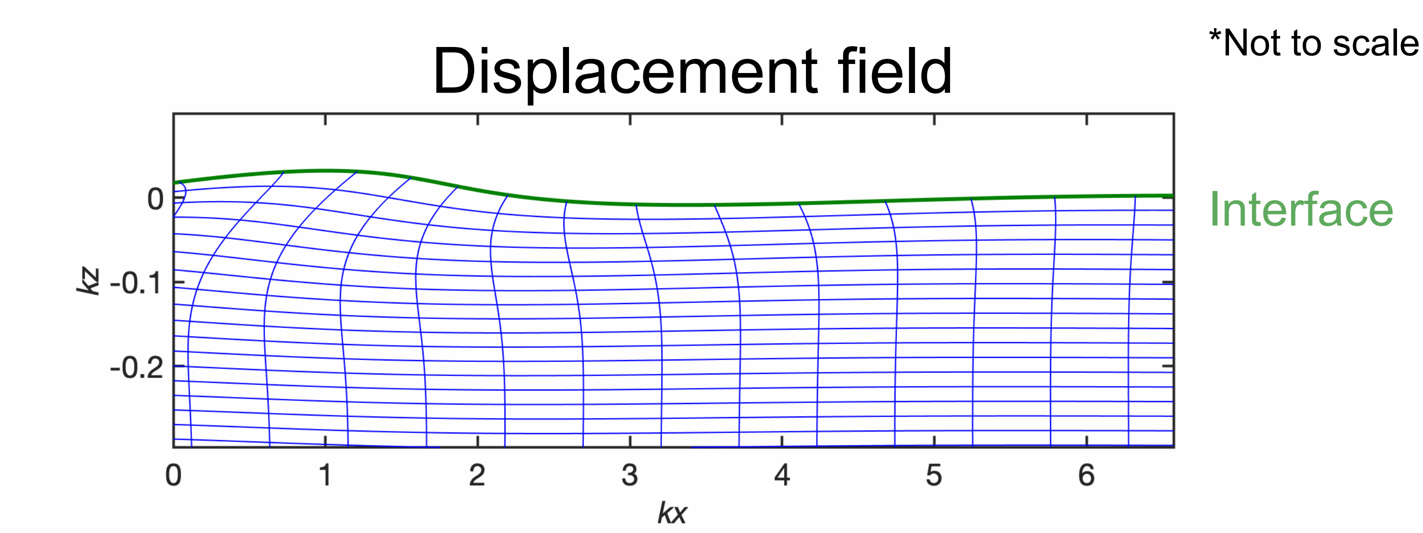
Diffusive-wave (superdiffusion)

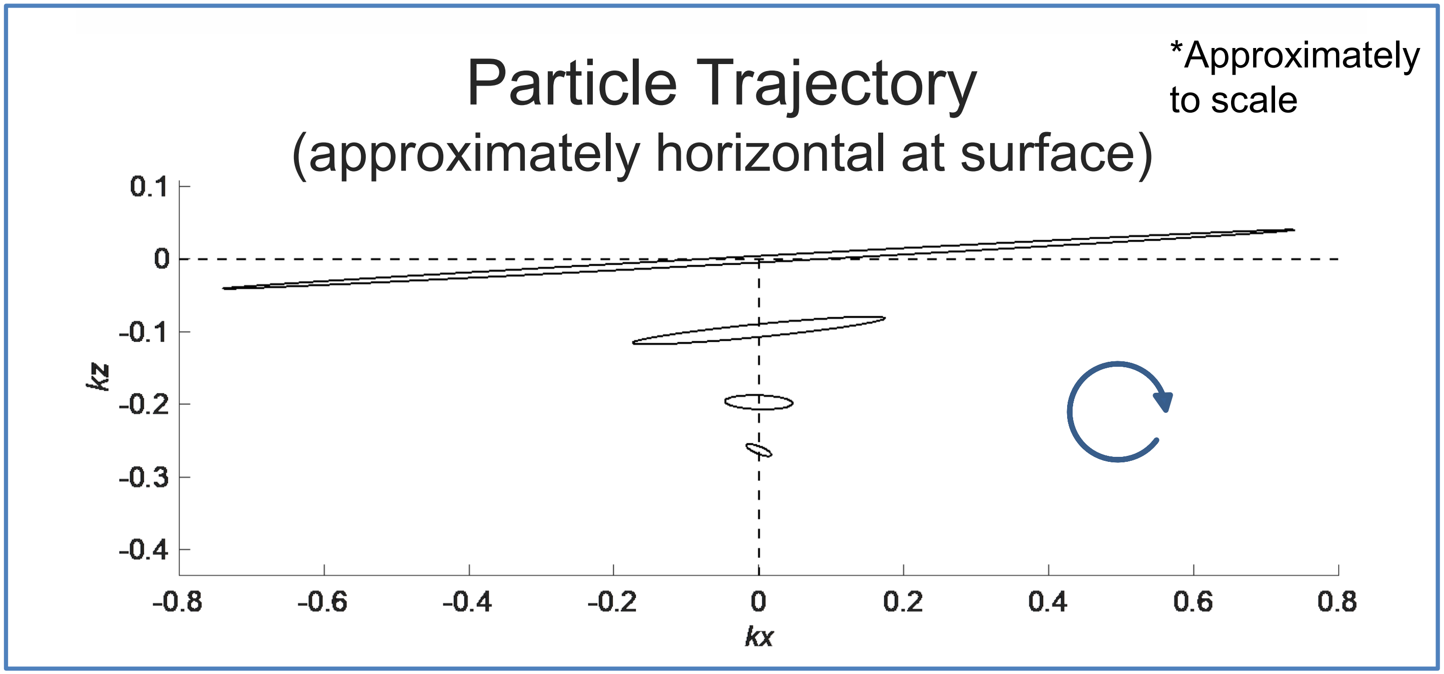
$$\frac{\partial^2 U}{\partial X^2} = \frac{\partial^{3/2} U}{\partial T^{3/2}}$$

Diffusion

$$\frac{\partial^2 U}{\partial X^2} = \frac{\partial U}{\partial T}$$

Visualizing Lucassen waves

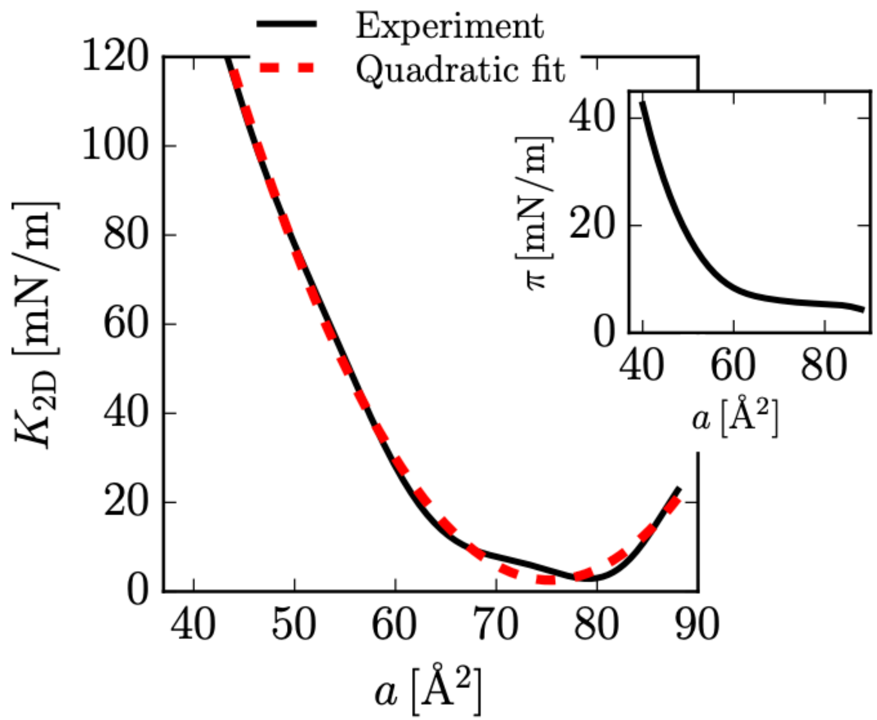




Incorporating nonlinearity: KSSN equation

Assume the surface elastic modulus varies with wave amplitude:

$$K_{2D}(u) \approx \kappa_0 + \kappa_1 \frac{\partial u}{\partial x} + \kappa_2 \left(\frac{\partial u}{\partial x}\right)^2$$



Substitution into the previous linear equation yields:

$$\left[\kappa_0 + \kappa_1 \frac{\partial u}{\partial x} + \kappa_2 \left(\frac{\partial u}{\partial x}\right)^2\right] \frac{\partial^2 u}{\partial x^2} = \sqrt{\rho\mu}\, \frac{\partial^{3/2} u}{\partial t^{3/2}}$$

Kappler, Shrivastava, Schneider, and Netz (Phys. Rev. Fluids 2017)

Approximate evolution equation

- KSSN compound wave equation:

$$\left[\kappa_0 + \kappa_1 \frac{\partial u}{\partial x} + \kappa_2 \left(\frac{\partial u}{\partial x}\right)^2\right] \frac{\partial^2 u}{\partial x^2} = \sqrt{\rho\mu} \frac{\partial^{3/2} u}{\partial t^{3/2}}$$

- New wave variable is interfacial compression:

$$\psi = -\frac{\partial u}{\partial x}$$

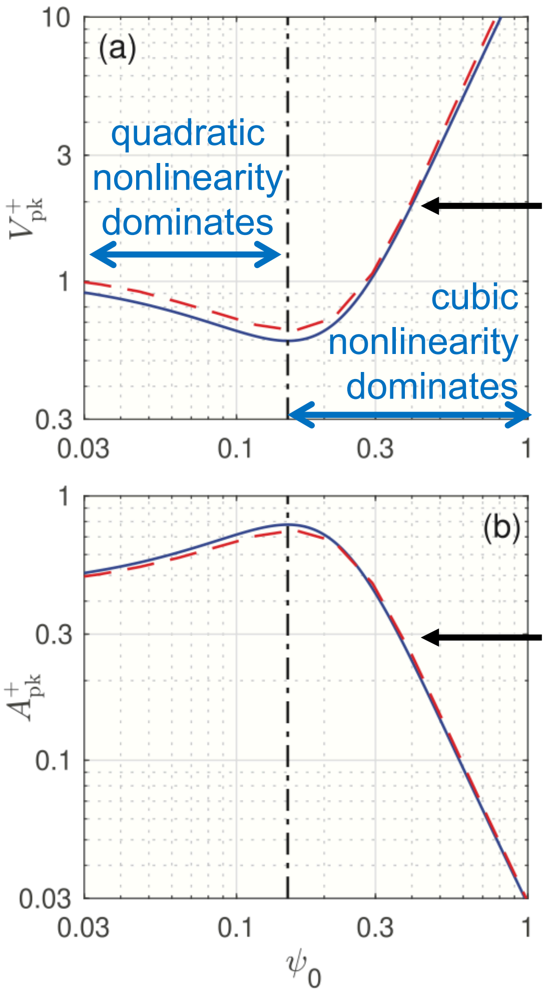
- Approximate evolution equation, computationally efficient:

$$\frac{\partial \psi}{\partial x} = -\frac{(\rho\mu/\kappa_0^2)^{1/4}}{1 - (\kappa_1/2\kappa_0)\psi + (\kappa_2/2\kappa_0)\psi^2} \frac{\partial^{3/4}\psi}{\partial t^{3/4}}$$

Simon, Cormack, and Hamilton (JASA 2021)

Threshold phenomenon

- Electrical nerve impulses exhibit a threshold phenomena
- A similar behavior is shown for Lucassen mechanical waves



Simon et al. (JASA 2021)

$$V_{pk}^{+} = 1 + \beta_2 \psi_0 + \beta_3 \psi_0^2$$

$$\beta_2 < 0, \;\; \beta_3 > 0$$

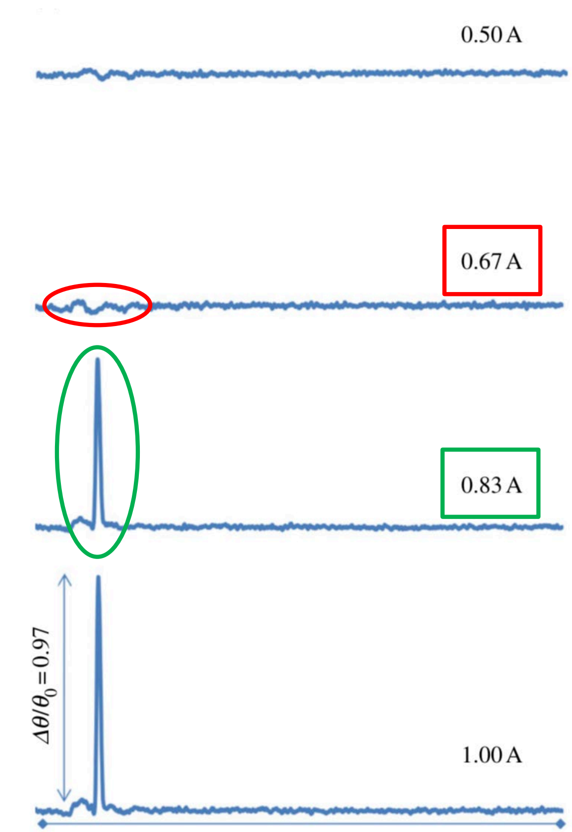
Velocity increases above threshold strain ψ_0

$$\psi = -\frac{\partial u}{\partial x}$$

Attenuation decreases above threshold strain ψ_0

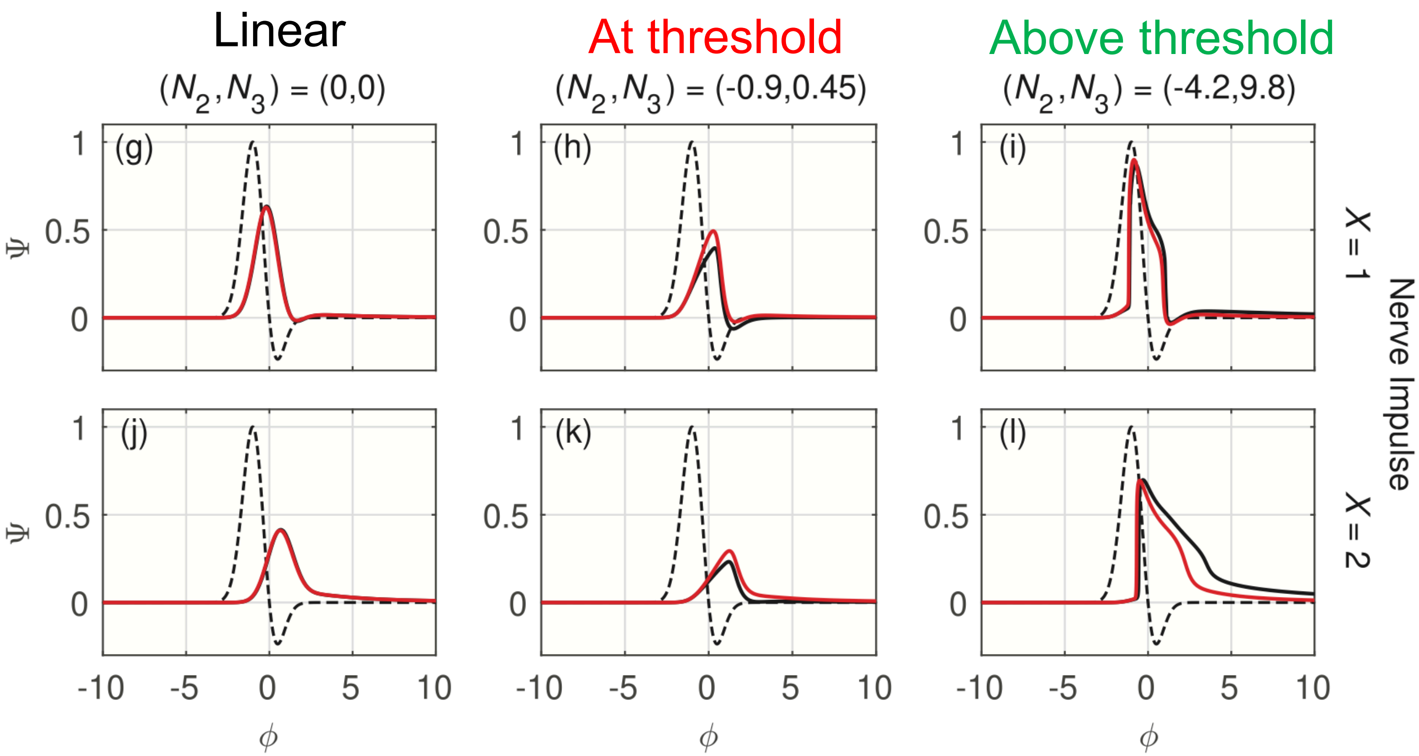
$$A_{pk}^{+} \propto 1/V_{pk}^{+}$$

Threshold effect observed in mechanical surface wave



Shrivastava and Schneider (J. R. Soc. Interface 2014)

Numerical simulations of threshold phenomenon



Simon et al. (JASA 2021)

A Paradox

A paradox

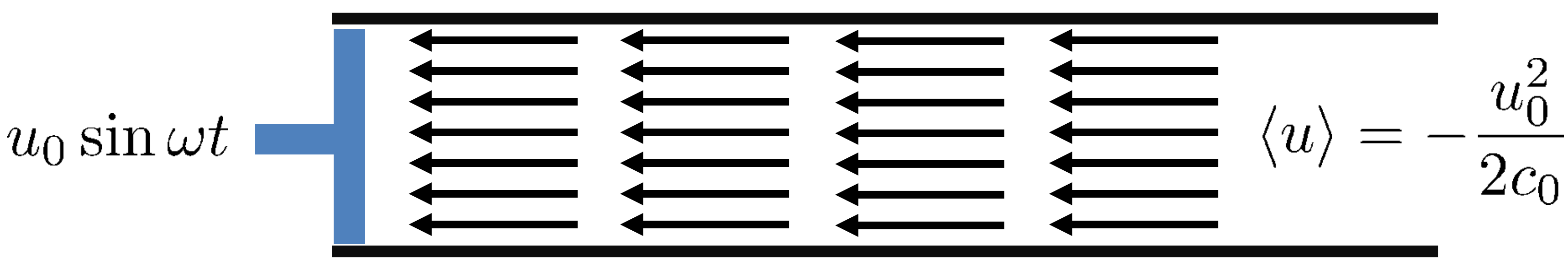
Continuity equation: $$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0$$

Take time average: $$\langle \rho u \rangle = 0$$

Let $\rho = \rho_0 + \rho'$: $$\langle u \rangle = -\frac{\langle \rho' u \rangle}{\rho_0}$$ (exact)

For any wave travelling to the right, $\langle \rho' u \rangle > 0$, and therefore the time-averaged velocity is negative: $\langle u \rangle < 0$.

For example, if in the linear approximation $u = u_0 \sin(\omega t - kx)$ and therefore $\rho' = (\rho_0 u_0 / c_0) \sin(\omega t - kx)$, then $\langle u \rangle = -(u_0)^2 / 2c_0$.

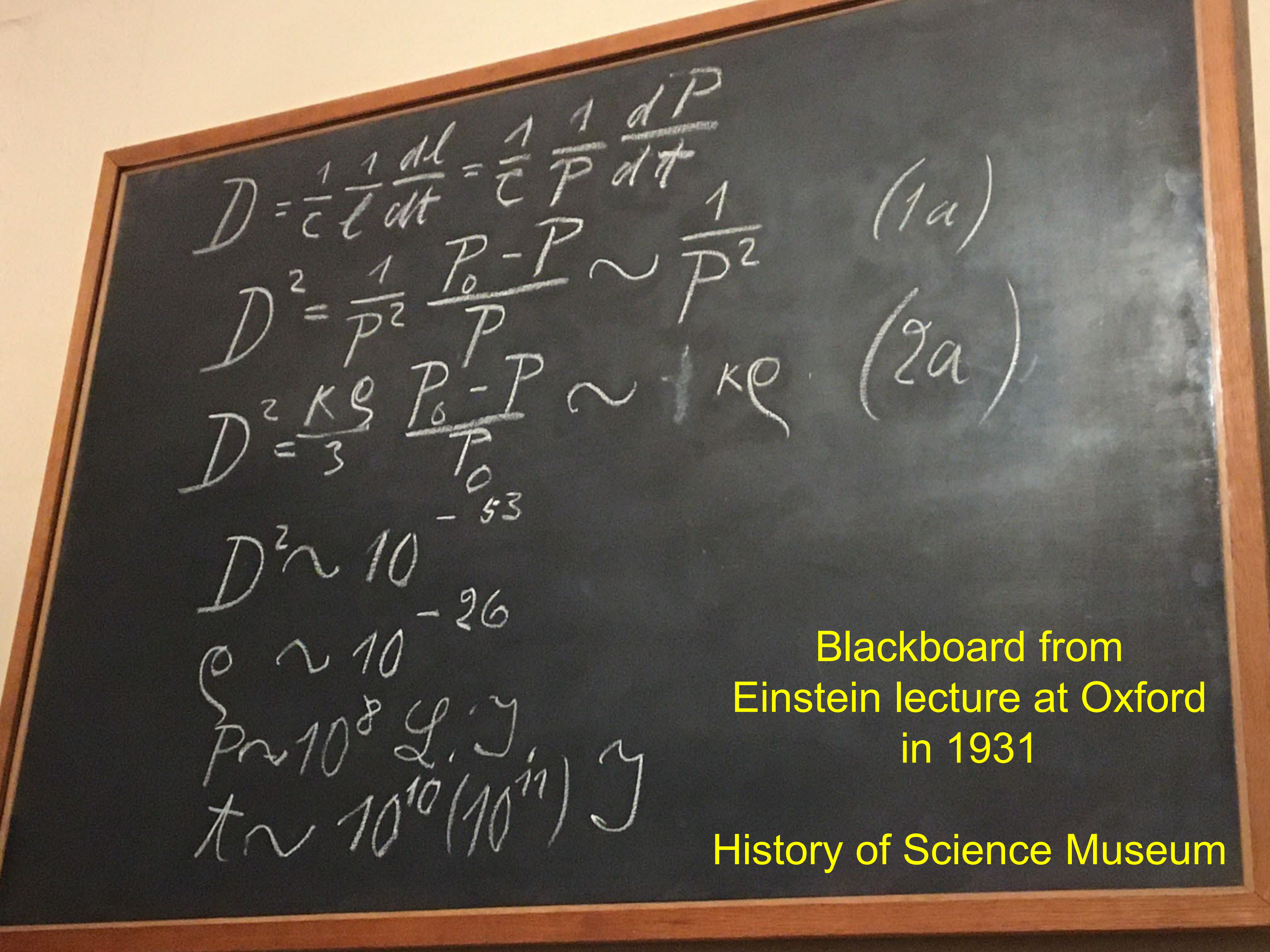


In memoriam—University of Texas at Austin

Yurii Ilinskii
1936—2019

Evgenia Zabolotskaya
1935—2020

David Blackstock
1930—2021



Blackboard from
Einstein lecture at Oxford
in 1931

History of Science Museum